

Gravito-electromagnetic analogies

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July 3, 2012

Abstract

We reexamine and further develop different gravito-electromagnetic (GEM) analogies found in the literature, and clarify the connection between them. Special emphasis is placed in two exact physical analogies: the analogy based on inertial fields from the so-called “1+3 formalism”, and the analogy based on tidal tensors. Both are reformulated, extended and generalized. We write in both formalisms the Maxwell and the full exact Einstein field equations with sources, plus the algebraic Bianchi identities, which are cast as the source-free equations for the gravitational field. New results within each approach are unveiled. The well known analogy between linearized gravity and electromagnetism in Lorentz frames is obtained as a limiting case of the exact ones. The formal analogies between the Maxwell and Weyl tensors, and the related issue of super-energy, are also discussed, and the physical insight from the tidal tensor formalism is seen to yield a suggestive interpretation of the phenomenon of gravitational radiation. The precise conditions under which a similarity between gravity and electromagnetism occurs are discussed, and we conclude by summarizing the main outcome of each approach.

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1 Introduction

This work has two main goals: one is to establish the connection between the several gravito-electromagnetic analogies existing in the literature, summarizing the main results and insights brought by each of them; the other is to further develop and extend some of these analogies.

In an earlier work by one of the authors [1, 2], a gravito-electromagnetic analogy based on tidal tensors was presented, and its relationship with 1) the well known analogy between linearized gravity and electromagnetism, 2) the mapping, via the Klein-Gordon equation, between ultrastationary spacetimes and magnetic fields in curved spacetimes, and 3) the formal analogies between the Weyl and Maxwell tensors (their decomposition in electric and magnetic parts, the quadratic scalar invariants they form, and the field equations they obey) was discussed.

Building up on the work in [2], another approach is herein added to the discussion: the exact analogy based on the fields of inertial forces, arising in the context of the 1+3 splitting of spacetime. This approach, which is herein reformulated and suitably generalized, is still not very well known,

but very far reaching. It is therefore important to understand how it relates with the other known analogies, and in particular with the (also exact) approach based on tidal tensors.

Each of the analogies discussed here are also further developed, and some new results within each of them are presented. We start in Sec. 2 by revisiting the analogy based on tidal tensors introduced in [1] (and partly reviewed in [3]). In [1] it was shown that there is an exact, physical gravito-electromagnetic analogy relating the so-called electric ($\mathbb{E}_{\alpha\beta}$) and magnetic ($\mathbb{H}_{\alpha\beta}$) parts of the Riemann tensor with the electromagnetic tidal tensors ($E_{\alpha\beta}$, $B_{\alpha\beta}$) defined therein: the “relative acceleration” between two nearby test particles, with the *same 4-velocity* U^α , in a gravitational field, is given by a contraction of $\mathbb{E}_{\alpha\beta}$ with the separation vector δx^β ; just like the “relative acceleration” between two nearby charged particles (with the same U^α and ratio q/m) is given by a contraction of the electric tidal tensor $E_{\alpha\beta}$ with δx^α . Moreover, the gravitational force exerted on a spinning particle is exactly given by a contraction of $\mathbb{H}_{\alpha\beta}$ with the spin vector S^α (the “gravitomagnetic dipole moment”), just like its electromagnetic counterpart is exactly given by a contraction of the magnetic tidal tensor $B_{\alpha\beta}$ with the magnetic dipole moment μ^α . Here $\mathbb{E}_{\alpha\beta}$, $\mathbb{H}_{\alpha\beta}$, $E_{\alpha\beta}$, $B_{\alpha\beta}$ are the tidal tensors as *measured by the test particles*. In this work we unveil another exact physical analogy involving $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$, manifest in the equations for the relative precession of spinning particles: both in electromagnetism and gravity, the relative precession of a test gyroscope/magnetic dipole relative to a system of axes attached to guiding gyroscopes/magnetic dipoles at a neighboring point, moving with *the same 4-velocity* U^α , is given by a contraction of $B_{\alpha\beta}/\mathbb{H}_{\alpha\beta}$ with the separation vector δx^β . This gives a physical interpretation of the tensors $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ complementary to the one described above. The expression for the differential precession of gyroscopes in terms of $\mathbb{H}_{\alpha\beta}$ was originally found in a recent work [102]; herein (Sec. 2.3) we re-derive this result through a different procedure (which, we believe, is more clear), and we obtain its electromagnetic analogue.

The analogy based on tidal tensors extends to the field equations of both theories; in [1] it was shown that by taking the traces and antisymmetric parts of the electromagnetic tidal tensors one obtains Maxwell equations, and performing the same operations in the gravitational tidal tensors leads to a strikingly similar set of equations, which turn out to be the temporal part of Einstein’s equations. Herein, building on that work, we extend this formalism to the full gravitational field equations (Einstein equations with sources plus the algebraic Bianchi identities); we show that they can be decomposed in a set of equations involving only the sources, the gravitoelectric $\mathbb{E}_{\alpha\beta}$ and gravitomagnetic $\mathbb{H}_{\alpha\beta}$ tidal tensors, plus a third spatial tensor $\mathbb{F}_{\alpha\beta}$, introduced by Bel [48], which has no electromagnetic analogue. More precisely, making a full 1+3 covariant splitting of the gravitational field equations, one obtains a subset of four equations (involving the temporal part of the curvature), which are formally similar to Maxwell equations written in this formalism (as found in [1]), *both being algebraic equations involving only tidal tensors and sources*, plus an additional pair of equations involving the purely spatial curvature (encoded in $\mathbb{F}_{\alpha\beta}$), which have no electromagnetic counterpart. In the case of vacuum, this latter part identically vanishes, and not only the full Einstein equations become algebraic equations for tidal tensors, just like the Maxwell’s, but also one can actually exactly obtain the former by simply replacing $\{E_{\alpha\beta}, B_{\alpha\beta}\} \rightarrow \{\mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta}\}$ in the latter. It is discussed how this approach, by bringing together the two theories in a single formalism, is especially suited for a transparent comparison between the two interactions, which reduces to comparing the tidal tensors of both sides (which is straightforward in this framework). Fundamental differences between the two interactions are encoded in the symmetries and time

projections of the tidal tensors (as already pointed out in [1]); herein we explore the consequences in terms of the worldline deviation of (monopole) test particles; and in the companion paper [4], in the dynamics of spinning multipole test particles.

In Sec. 3, we discuss another exact gravito-electromagnetic analogy, the one drawing a parallelism between spatial inertial forces (described by the GEM fields) and the electromagnetic fields. GEM fields are best known from linearized theory, e.g. [23, 98, 97, 64, 24]; less well known is the exact “Quasi-Maxwell” analogy, based on the inertial fields that arise in the 1+3 splitting of stationary spacetimes introduced by Landau-Lifshitz [77], and further worked out by many authors [15, 14, 16, 93, 94, 92]. Even less known is the existence of an exact formulation applying to *arbitrary* spacetimes [19, 20], which considers an arbitrary timelike congruence and does a general 1+3 splitting in terms of a time direction parallel to the corresponding observers’ worldlines, and the local rest spaces orthogonal to them. In this latter framework, the gravito-electromagnetic analogy (in the sense of a one to one correspondence) is, for most effects, only recovered for stationary spacetimes (exceptions are the case of the spin precession, and the hidden momentum of a spinning particle discussed in [4]). Herein we reformulate this approach in a slightly more general form, in the sense that we use an arbitrary (not the congruence adapted) frame, which splits the gravitomagnetic field \vec{H} in its two constituent parts, of different mathematical origin: the vorticity $\vec{\omega}$ of the observer congruence and the rotation $\vec{\Omega}$ of the local tetrads (associated to each local observer) relative to local Fermi-Walker transport. This degree of generality is of use both in the companion paper [4], and in [63]. It also allows, for instance, one to realize that there is a distinction between the gravitomagnetic effects detected in the analysis of the LAGEOS Satellites data [41] (and presently under scrutiny in the ongoing LARES mission [43]), and the one measured by the Gravity Probe B experiment. An important result of this approach is an exact expression for the geodesic equation, written in the language of GEM fields, that is fully general (valid for arbitrary fields, and formulated in terms of an arbitrary frame); the approximate GEM expressions from the Post-Newtonian and linearized theory in the literature, as well as the exact expressions of the Quasi-Maxwell formalism for stationary spacetimes, are just special cases of this general equation. The exact, mathematically rigorous formulation yields a powerful formalism with a very broad spectrum of applications; and an adequate account of some subtleties involved which are overlooked in the more common linear approaches. Indeed, the inertial GEM fields in this formulation — the gravitoelectric field \vec{G} , which is but *minus* the acceleration of the congruence’s observers, and the gravitomagnetic field $\vec{H} = \vec{\omega} + \vec{\Omega}$ — can be regarded as the general, exact form of the GEM 3-vectors fields of the usual linearized theory, which (in the way they are usually presented) are somewhat naively defined from the temporal components of the metric tensor (drawing a parallelism with the electromagnetic potentials), without making transparent their status as artifacts of the reference frame, and in particular their relation with the kinematical quantities associated to the observer’s congruence. Problems which in the approximate descriptions usually end up being treated superficially, overlooking the complicated underlying problem, are the effects concerning gyroscope “precession” relative to the “distant stars” (such as the Lense-Thirring and the geodetic precessions) — the question arising: how can one talk about the “precession” of a local gyroscope relative to the distant stars, as it amounts to comparing systems of vectors at different points in a curved spacetime? Such comparison can be done in a certain class of spacetimes, and the mathematical basis for it is discussed in Secs. 3.1 and 3.3.

We clarify the relationship between these inertial fields and the tidal tensors of Sec. 2 (a particularly important result in the context of this work). For stationary spacetimes, and frames adapted to stationary observers (Quasi-Maxwell regime), we express, as in Sec. 2, the Maxwell equations, and the full Einstein equations and algebraic Bianchi identities, in this formalism. Again, a set of four equations — the time-time and time-space components of the gravitational equations — are produced which exhibit a striking formal similarity with their electromagnetic counterparts; and there is a fifth equation, the space-space component of Einstein’s equations, which has no gravitational analogue. We show also that an exact GEM analogy relating the force on a gyroscope with the force on a magnetic dipole exists also in this formalism. In an earlier work by one of the authors [14], the force exerted on a gyroscope, at rest with respect to a rigid frame in a stationary spacetime was written in terms of the GEM fields associated to that frame; herein we obtain the electromagnetic counterpart: the force exerted on a magnetic dipole at rest in a rigid, arbitrarily accelerated frame, in terms of the electromagnetic fields associated to that frame, and show there is a one to one correspondence with the gravitational analogue.

In Sec. 4 we discuss a special class of spacetimes admitting global rigid geodesic congruences, the “ultra-stationary” spacetimes. They have interesting properties in the context of GEM, which were discussed in [1]: they are exactly mapped, via the Klein-Gordon equation, to magnetic fields in curved spacetimes, and have a linear gravitomagnetic tidal tensor, matching the electromagnetic analogue. Herein we revisit those spacetimes in the framework of the GEM inertial fields of Sec. 3, which sheds light on their properties: these metrics are characterized by a vanishing gravitoelectric field and a non-vanishing gravitomagnetic field which is *linear in the metric tensor*; these two properties together explain the above mentioned exact mapping, and the linearity of the gravitomagnetic tidal tensor. We also interpret the non-vanishing of gravitoelectric tidal tensor, a question left open in [1].

In Sec. 5 we explain the relation between the exact approach based in the inertial GEM fields of Sec. 3, and the popular gravito-electromagnetic analogy based on linearized theory, e.g. [23, 98, 64, 97]. The latter is obtained as a special limit of the exact equations of the former. Taking this route allows, as explained above, to have a clearer account of the physical meaning of the GEM fields and other kinematical quantities involved. It is also a procedure for obtaining the field equations in terms of (physically meaningful) GEM fields that does not rely on choosing the harmonic gauge condition and its inherent subtleties (that have been posing some difficulties in the literature, see e.g. [97, 119, 64, 2]). The usual expression for the force on a gyroscope in the literature (e.g. [7, 24, 23]) is also seen to be a (very) limiting case of the exact equation given in the tidal tensor formalism of Sec. 2.

In Sec. 6 we discuss the analogy between the electric ($\mathcal{E}_{\alpha\beta}$) and magnetic ($\mathcal{H}_{\alpha\beta}$) parts of the Weyl tensor, and the electromagnetic fields E^α and B^α . As already discussed in [2], this is a purely formal analogy, as it deals with tidal tensors in the gravitational side, and vector fields (not tidal tensors, which are one order higher in differentiation) of electromagnetism. One of the major results of this approach is that the differential Bianchi identities (the higher order field equations), when expressed in terms of $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$, exhibit some formal similarities with the Maxwell equations written in terms of E^α , B^α , measured with respect to an arbitrarily shearing, rotating and expanding congruence. In the case of vacuum, in the linear regime, they become Matte’s equations — a set of equations formally similar to Maxwell’s equations in a Lorentz frame, only with the gravitational

tidal tensors ($\mathcal{E}_{ij} = \mathbb{E}_{ij}$, $\mathcal{H}_{ij} = \mathbb{H}_{ij}$) in place of the electromagnetic fields. We arrive at a suggestive interpretation of gravitational radiation as a pair of traveling orthogonal tidal tensors, propagating by mutually inducing each other (just like \vec{E} and \vec{B} in the electromagnetic waves), and whose effects on test particles are readily established via the tidal tensor approach of Sec. 2. Such “tidal tensor interpretation” of the gravitational waves has not been put forth in the literature, possibly for two reasons: one is that in the literature dealing with the higher order field equations the physical interpretation of $\mathcal{H}_{\alpha\beta}$ appears as an unanswered question [84, 87, 56, 82, 83, 32]; the other is that in the literature dealing with the interaction of gravitational waves with spinning particles [72, 71, 73, 74], neither the waves are written in tidal tensor form (but instead in the more usual equations for the propagation of the metric perturbations), nor the force on a spinning particle is cast in terms of $\mathcal{H}_{\alpha\beta}$, since the Mathisson-Pirani condition is not employed (the latter is necessary in order to make $\mathcal{H}_{\alpha\beta}$ explicitly appear in the equations, see [4]). This analogy has been used to address the problem of the definition of local quantities for the gravitational field analogous to the electromagnetic field energy and Poynting vector, which has motivated the definition of the Bel-Robinson tensor (the so-called “superenergy” tensor). The viewpoint that gravitational waves should be thought of as carrying superenergy, not energy, fits well in the interpretation above, and with what it tells us about the interaction of waves with multipole test particles.

1.1 Notation and conventions

1. *Signature and signs.* We use the signature $-+++$; $\epsilon_{\alpha\beta\sigma\gamma} \equiv \sqrt{-g}[\alpha\beta\gamma\delta]$ denotes the Levi-Civita tensor, and we follow the orientation $[1230] = 1$ (i.e., in flat spacetime $\epsilon_{1230} = 1$). $\epsilon_{ijk} \equiv \epsilon_{ijk0}$ is the 3-D alternating tensor. $\star \equiv$ Hodge dual.
2. *Time and space projectors.* $(\top^u)^\alpha_\beta \equiv -u^\alpha u_\beta$, $(h^u)^\alpha_\beta \equiv u^\alpha u_\beta + g^\alpha_\beta$ are, respectively, the projectors parallel and orthogonal to a unit time-like vector u^α ; may be interpreted as the time and space projectors in the local rest frame of an observer of 4-velocity u^α . $\langle\alpha\rangle$ denotes the index of a spatially projected tensor: $A^{\langle\alpha\rangle\beta\dots} \equiv (h^u)^\alpha_\beta A^{\mu\beta\dots}$.
3. $\rho_c = -j^\alpha U_\alpha$ and j^α are, respectively, the charge density and current 4-vector; $\rho = T_{\alpha\beta} U^\alpha U^\beta$ and $J^\alpha = -T^\alpha_\beta U^\beta$ are the mass/energy density and current (quantities measured by the observer of 4-velocity U^α); $T_{\alpha\beta} \equiv$ energy-momentum tensor.
4. $S^\alpha \equiv$ spin 4-vector; $\mu^\alpha \equiv$ magnetic dipole moment, defined as vectors whose components in the particle’s proper frame are $S^\alpha = (0, \vec{S})$, $\vec{\mu} = (0, \vec{\mu})$.
5. *Tensors resulting from a measurement process.* $(A^u)^{\alpha_1\dots\alpha_n}$ denotes the tensor \mathbf{A} as measured by an observer $\mathcal{O}(u)$ of 4-velocity u^α . For example, $(E^u)^\alpha \equiv F^\alpha_\beta u^\beta$, $(E^u)_{\alpha\beta} \equiv F_{\alpha\gamma;\beta} u^\gamma$ and $(\mathbb{E}^u)_{\alpha\beta} \equiv R_{\alpha\nu\beta\mu} u^\nu u^\mu$ denote, respectively, the electric field, electric tidal tensor, and gravito-electric tidal tensor as measured by $\mathcal{O}(u)$. Analogous forms apply to their magnetic/gravitomagnetic counterparts.

For 3-vectors we use notation $\vec{A}(u)$; for example, $\vec{E}(u)$ denotes the electric 3-vector field as measured by $\mathcal{O}(u)$ (i.e., the space part of $(E^u)^\alpha$, written in a frame where $u^i = 0$). Often we drop the superscript (e.g. $(E^U)^\alpha \equiv E^\alpha$), or the argument of the 3-vector: $\vec{E}(U) \equiv \vec{E}$, when the meaning is clear.

6. *Electromagnetic field.* The Maxwell tensor $F^{\alpha\beta}$ and its Hodge dual $\star F^{\alpha\beta} \equiv \epsilon_{\mu\nu}^{\alpha\beta} F_{\mu\nu}/2$ decompose in terms of the electric $(E^u)^\alpha \equiv F^\alpha_\beta u^\beta$ and magnetic $(B^u)^\alpha \equiv \star F^\alpha_\beta u^\beta$ fields measured by an observer of 4-velocity u^α as

$$F_{\alpha\beta} = 2u_{[\alpha}(E^u)_{\beta]} + \epsilon_{\alpha\beta\gamma\delta} u^\delta (B^u)^\gamma \quad (\text{a}) \quad \star F_{\alpha\beta} = 2u_{[\alpha}(B^u)_{\beta]} - \epsilon_{\alpha\beta\gamma\sigma} u^\sigma (E^u)^\gamma \quad (\text{b}) \quad (1)$$

2 The gravito-electromagnetic analogy based on tidal tensors

The rationale behind the tidal tensor gravito-electromagnetic analogy is to make a comparison between the two interactions based on physical forces present in both theories. The electromagnetic Lorentz force has no physical counterpart in gravity, as monopole point test particles in a gravitational field move along geodesics, without any force being exerted on them. In this sense the analogy drawn in Sec. 3.2, between Eqs. (46)-(45), is a comparison of a physical electromagnetic force to an artifact of the reference frame. Tidal forces, by their turn, are covariantly present in both theories, and their mathematical description in terms of objects called “tidal tensors” is the basis of this approach. Tidal forces manifest themselves in essentially two basic effects: the relative acceleration of two nearby monopole test particles, and in the net force exerted on dipoles. These notions of multipole moments arise from a description of the *test* bodies in terms of the fields they *would* produce. In electromagnetism is the multipole expansion of the current density vector $j^\alpha = (\rho_c, \vec{j})$, rigorously established in [9], and well known in textbooks as the moments of the charge and current densities. In gravity is the moments of the energy momentum tensor $T_{\alpha\beta}$, the so called [8] “gravitational skeleton”, of which only the moments of the current density $J^\alpha = -T^\alpha_\beta U^\beta$ have an electromagnetic counterpart. Monopole particles in the context of electromagnetism are those whose only non-vanishing moment is the total charge; dipole particles are particles with nonvanishing electric and magnetic dipole moments (i.e., respectively the dipole moments of ρ_c and \vec{j}); see [9] and companion paper [4] for precise definitions of these moments. Monopole particles in gravity are particles whose only non-vanishing moment of $T^{\alpha\beta}$ is the mass, and correspond to the usual notion of *point test particle*, that moves along geodesics. There is no gravitational analogue to the intrinsic electric dipole, as there are no negative masses; but in the multipole scheme, there is an analogue to the magnetic dipole moment, which is the “intrinsic” angular momentum (i.e. the angular momentum about the particle’s center of mass), usually dubbed spin vector/tensor. Sometimes it will also be dubbed herein, for obvious reasons, “gravitomagnetic dipole moment”. A particle possessing only pole-dipole gravitational moments corresponds to the notion of an ideal gyroscope. We have thus two physically analogous effects suited to compare gravitational and electromagnetic tidal forces: worldline deviation of nearby monopole test particles, and the force exerted on magnetic dipoles/gyroscopes. An exact gravito-electromagnetic analogy, summarized in Table 1, emerges from that comparison.

Eqs. (1.1) are the worldline deviations for nearby test particles with the same¹ tangent vector (and the same ratio charge/mass in the electromagnetic case), separated by the infinitesimal vector

¹We would like to emphasize this point which, even today, is not clear in the literature. Eqs. (1.1) apply only to the instant where the two particles have the same (or infinitesimally close, in the gravitational case) tangent vector. This seems to be well understood in the case of the electromagnetic deviation (1.1a), but is not so with the geodesic deviation equation (1.1a); which is presented in some literature as applying to the case where the deviation velocity is not zero [28, 120, 121], there being claims that it is an arbitrary initial condition [28]. That is not the case: for both electromagnetism and gravity, in the more general case that the velocity of the two particles is not

Table 1: The gravito-electromagnetic analogy based on tidal tensors.

Electromagnetism	Gravity
Worldline deviation:	Geodesic deviation:
$\frac{D^2 \delta x^\alpha}{d\tau^2} = \frac{q}{m} E^\alpha_\beta \delta x^\beta, \quad E^\alpha_\beta \equiv F^\alpha_{\mu;\beta} U^\mu \quad (1.1a)$	$\frac{D^2 \delta x^\alpha}{d\tau^2} = -\mathbb{E}^\alpha_\beta \delta x^\beta, \quad \mathbb{E}^\alpha_\beta \equiv R^\alpha_{\mu\beta\nu} U^\mu U^\nu \quad (1.1b)$
Force on magnetic dipole:	Force on gyroscope:
$F^\beta_{EM} = B^\beta_\alpha \mu^\alpha, \quad B^\alpha_\beta \equiv \star F^\alpha_{\mu;\beta} U^\mu \quad (1.2a)$	$F^\beta_G = -\mathbb{H}^\beta_\alpha S^\alpha, \quad \mathbb{H}^\alpha_\beta \equiv \star R^\alpha_{\mu\beta\nu} U^\mu U^\nu \quad (1.2b)$
Differential precession of magnetic dipoles:	Differential precession of gyroscopes:
$\delta \Omega^i_{EM} = \mathbb{H}^i_\beta \delta x^\beta \quad (1.8a)$	$\delta \Omega^i_G = -\sigma B^i_\beta \delta x^\beta \quad (1.8b)$
Maxwell Source Equations	Einstein Equations
$F^{\alpha\beta}_{;\beta} = 4\pi J^\beta$	$R_{\mu\nu} = 8\pi (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\alpha_\alpha)$
• Time Projection:	• Time-Time Projection:
$E^\alpha_\alpha = 4\pi \rho_c \quad (1.3a)$	$\mathbb{E}^\alpha_\alpha = 4\pi (2\rho + T^\alpha_\alpha) \quad (1.3b)$
• Space Projection:	• Time-Space Projection:
$B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^\gamma - 2\pi \epsilon_{\alpha\beta\sigma\gamma} j^\sigma U^\gamma \quad (1.6a)$	$\mathbb{H}_{[\alpha\beta]} = -4\pi \epsilon_{\alpha\beta\sigma\gamma} J^\sigma U^\gamma \quad (1.6b)$
	• Space-Space Projection:
<i>No electromagnetic analogue</i>	$\mathbb{F}^\alpha_\beta + \mathbb{E}^\alpha_\beta - \mathbb{F}^\sigma_\sigma h^\alpha_\beta = 8\pi \left[\frac{1}{2} T^\gamma_\gamma h^\alpha_\beta - T^{\langle\alpha}_{\langle\beta} \rangle \right] \quad (1.7)$
Bianchi Identity	Algebraic Bianchi Identity
$\star F^{\alpha\beta}_{;\beta} = 0 \quad (\Leftrightarrow F_{[\alpha\beta;\gamma]} = 0)$	$\star R^{\gamma\alpha}_{\gamma\beta} = 0 \quad (\Leftrightarrow R_{[\alpha\beta\gamma]\delta} = 0)$
• Time Projection:	• Time-Time (or Space-Space) Proj:
$B^\alpha_\alpha = 0 \quad (1.5a)$	$\mathbb{H}^\alpha_\alpha = 0 \quad (1.5b)$
• Space Projection:	• Space-Time Projection:
$E_{[\alpha\beta]} = \frac{1}{2} F_{\alpha\beta;\gamma} U^\gamma \quad (1.4a)$	$\mathbb{E}_{[\alpha\beta]} = 0 \quad (1.4b)$
	• Time-Space Projection:
<i>No electromagnetic analogue</i>	$\mathbb{F}_{[\alpha\beta]} = 0$

δx^α . They tell us that the so-called electric part of the Riemann tensor $\mathbb{E}^\alpha_\beta \equiv R^\alpha_{\mu\beta\nu} U^\mu U^\nu$ plays the same physical role as the tensor $E_{\alpha\beta} \equiv F_{\alpha\gamma;\beta} U^\gamma$ in the electromagnetic worldline deviation (1.1a). $E_{\alpha\beta}$ measures the tidal effects produced by the electric field $E^\alpha = F^\alpha_\gamma U^\gamma$ as measured by the test particle of 4-velocity U^α . We can define it as a covariant derivative of the electric field as measured in the inertial frame momentarily comoving with the particle: $E_{\alpha\beta} = E_{\alpha;\beta}|_{U=\text{const.}}$. Hence we dub it “electric tidal tensor”, and its gravitational

infinitesimally close, the deviation equations include more terms (which depend on both particle’s 4-velocity, thus in their relative velocity); the relative acceleration is not, in either case, given by a simple contraction of a tidal tensor with a separation vector, see [1, 2, 29]. A more detailed discussion of this important issue will be presented elsewhere.

counterpart “gravitoelectric tidal tensor”. Eqs (1.2) are respectively, the electromagnetic force on a magnetic dipole [4], and the Mathisson-Papapetrou-Pirani [8, 10, 11, 4] equation for the gravitational force exerted on a gyroscope. They tell us that the magnetic part of the Riemann tensor $\mathbb{H}^\alpha_\beta \equiv \star R^\alpha_{\mu\beta\nu} U^\mu U^\nu$ plays in the gravitational force (1.2b) the same physical role as the tensor $B_{\alpha\beta} \equiv \star F_{\alpha\gamma;\beta} U^\gamma$ in the electromagnetic force (1.2a). $B_{\alpha\beta}$ measures the tidal effects produced by the magnetic field $B^\alpha = \star F^\alpha_\gamma U^\gamma$ *as measured by the particle* of 4-velocity U^α ; for this reason we dub it “magnetic tidal tensor”, and its gravitational analogue $\mathbb{H}_{\alpha\beta}$ “gravitomagnetic tidal tensor”.

2.1 Tidal tensor formulation of Maxwell and Einstein equations

Taking time and space projections Maxwell’s and Einstein’s can be expressed in tidal tensor formalism; that makes explicit a striking aspect of the analogy: Maxwell’s equations (the source equations plus Bianchi identity) may be cast as a set of *algebraic* equations involving only tidal tensors and source terms (the charge current 4-vector); and Einstein’s equations (the source equations plus the algebraic Bianchi identity) consist of two parts: i) a temporal part that is formally very similar to Maxwell’s equations and are likewise algebraic equations involving only the gravitoelectric and gravitomagnetic tidal tensors and sources (the mass-energy current vector), and ii) a space-space part which involves a third spatial tensor and has no electromagnetic analogue. That is what we are going to show next. For that we first introduce the time and space projectors with respect to a unit time-like vector U^α (i.e., the projectors parallel and orthogonal to U^α):

$$\top^\alpha_\beta \equiv (\top^U)^\alpha_\beta = -U^\alpha U_\beta; \quad h^\alpha_\beta \equiv (h^U)^\alpha_\beta = U^\alpha U_\beta + \delta^\alpha_\beta. \quad (2)$$

A vector A^α can be split in its time and space projections with respect to U^α ; and arbitrary rank n tensor can be completely decomposed taking time and space projections in each of its indices (e.g. [19]):

$$A^\alpha = \top^\alpha_\beta A^\beta + h^\alpha_\beta A^\beta; \quad A^{\alpha_1 \dots \alpha_n} = \left(\top^{\alpha_1}_{\beta_1} + h^{\alpha_1}_{\beta_1} \right) \dots \left(\top^{\alpha_n}_{\beta_n} + h^{\alpha_n}_{\beta_n} \right) A^{\beta_1 \dots \beta_n}. \quad (3)$$

Instead of using h^μ_σ , one can also, if convenient, spatially project an index of a tensor $A^{\sigma \dots}$ contracting it with the spatial 3-form $\epsilon_{\alpha\beta\sigma\gamma} U^\gamma$; for instance, for the case of vector A^σ , one obtains the spatial 2-form $\epsilon_{\alpha\beta\sigma\gamma} U^\gamma A^\sigma = \star A_{\alpha\beta\gamma} U^\gamma$, which contains precisely the same information as the spatial vector $A^\mu h^\mu_\sigma \equiv A^{(\sigma)}$ (the former is the spatial dual of the latter). New contraction with $\epsilon^{\alpha\beta}_{\mu\nu} U^\nu$ yields $A^{(\sigma)}$ again. Indeed we may write

$$h^\mu_\sigma = \epsilon^{\alpha\beta\mu\nu} U_\nu \epsilon_{\alpha\beta\sigma\gamma} U^\gamma.$$

Another very useful relation is the following. The space projection $h^\mu_\alpha h^\nu_\beta F_{\mu\nu} \equiv F_{\langle\alpha\rangle\langle\beta\rangle}$ of a 2-form $F_{\alpha\beta} = F_{[\alpha\beta]}$ is equivalent to the tensor $\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} U^\nu = \star F_{\mu\nu} U^\nu$ (i.e., spatially projecting $F_{\alpha\beta}$ is equivalent to time-projecting its Hodge dual). We have:

$$F_{\langle\alpha\rangle\langle\beta\rangle} = \epsilon_{\mu\alpha\beta\lambda} U^\lambda \epsilon^\mu_{\nu\sigma\delta} U^\nu F^{\sigma\delta} = \epsilon_{\mu\alpha\beta\lambda} U^\lambda \star F^\mu_\nu U^\nu \quad (4)$$

Now let $F_{\gamma_1 \dots \gamma_n \alpha \beta \delta_1 \dots \delta_m} = F_{\gamma_1 \dots \gamma_n [\alpha \beta] \delta_1 \dots \delta_m}$, be some tensor antisymmetric in the pair α, β ; an equality similar to the first one above applies:

$$F_{\gamma_1 \dots \gamma_n \langle\alpha\rangle\langle\beta\rangle \delta_1 \dots \delta_m} = \epsilon_{\mu\alpha\beta\lambda} U^\lambda \epsilon^\mu_{\nu\sigma\delta} U^\nu F_{\gamma_1 \dots \gamma_n}{}^{\sigma\delta}{}_{\delta_1 \dots \delta_m} \quad (5)$$

2.1.1 Maxwell's equations

Maxwell equations are cast in tensor form by the pair of equations:

$$F^{\alpha\beta}{}_{;\beta} = 4\pi j^\alpha \quad (a); \quad \star F^{\alpha\beta}{}_{;\beta} = 0 \quad (b). \quad (6)$$

being (6a) the Maxwell source equations and the source free Eq. (6b), equivalent to $F_{[\alpha\beta;\gamma]} = 0$, is commonly called the Bianchi identity for $F_{\alpha\beta}$. These equations can be expressed in terms of tidal tensors using the decompositions

$$F_{\alpha\beta;\gamma} = 2U_{[\alpha}E_{\beta]\gamma} + \epsilon_{\alpha\beta\mu\sigma}U^\sigma B^\mu{}_\gamma \quad (7)$$

$$\star F_{\alpha\beta;\gamma} = 2U_{[\alpha}B_{\beta]\gamma} - \epsilon_{\alpha\beta\mu\sigma}U^\sigma E^\mu{}_\gamma \quad (8)$$

These expressions are obtained decomposing the tensors $F_{\alpha\beta;\gamma}$ and $\star F_{\alpha\beta;\gamma}$ in their time and space projections in the first two indices, using Eq. (4) to project spatially. Taking the time projection of (6a), we obtain Eq. (1.3a) of Table 1; taking the space projection by contracting with the spatial 3-form $\epsilon_{\mu\nu\alpha\sigma}U^\sigma$, yields Eq. (1.6a). The same procedure applied to Eq. (6b), yields Eqs. (1.4a) and (1.5a) as time and space projections, respectively.

Hence, in this formalism, Maxwell's equations are cast as the Eqs. of the traces and antisymmetric parts of the electromagnetic tidal tensors; and they involve only tidal tensors and sources, which is easily seen substituting the decompositions (7)-(8) in Eqs (1.4a) and (1.6a), leading to the equivalent set:

$$E^\alpha{}_\alpha = 4\pi\rho_c \quad (9)$$

$$E_{[\alpha\beta]} = U_{[\alpha}E_{\beta]\gamma}U^\gamma + \frac{1}{2}\epsilon_{\alpha\beta\mu\sigma}U^\sigma B^{\mu\gamma}U_\gamma \quad (10)$$

$$B^\alpha{}_\alpha = 0 \quad (11)$$

$$B_{[\alpha\beta]} = U_{[\alpha}B_{\beta]\gamma}U^\gamma - \frac{1}{2}\epsilon_{\alpha\beta\mu\sigma}U^\sigma E^{\mu\gamma}U_\gamma - 2\pi\epsilon_{\alpha\beta\sigma\gamma}j^\sigma U^\gamma \quad (12)$$

The pair of Eqs. (10) and (12) can be condensed in the equivalent pair

$$\epsilon^{\beta\gamma}{}_{\alpha\delta}U^\delta E_{[\gamma\beta]} = -B_{\alpha\beta}U^\beta; \quad (a) \quad \epsilon^{\beta\gamma}{}_{\alpha\delta}U^\delta B_{[\gamma\beta]} = E_{\alpha\beta}U^\beta + 4\pi j_\alpha \quad (b) \quad (13)$$

In a Lorentz frame, since $U^\alpha{}_{;\beta} = U^\alpha{}_{,\beta} = 0$, we have $E_{\gamma\beta} = E_{\gamma;\beta}$, $B_{\gamma\beta} = B_{\gamma;\beta}$; and (using $U^\alpha = \delta^\alpha_0$) Eqs. (13) can be written in the familiar vector forms $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$ and $\nabla \times \vec{B} = \partial\vec{E}/\partial t + 4\pi\vec{j}$, respectively. Likewise, Eqs. (9) and (11) reduce in this frame to the familiar forms $\nabla \cdot \vec{E} = 4\pi\rho_c$ and $\nabla \cdot \vec{B} = 0$, respectively.

2.1.2 Einstein's equations

Equations (14a) below are the Einstein source equations for the gravitational field; Eqs (14b) are the algebraic Bianchi identity, equivalent to $R_{[\alpha\beta\gamma]\delta} = 0$:

$$R^\gamma{}_{\alpha\gamma\beta} \equiv R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T^\gamma{}_\gamma \right) \quad (a); \quad \star R^\gamma{}_{\gamma\beta}{}^\alpha = 0 \quad (b). \quad (14)$$

In order to express these equations in the tidal tensor formalism we will decompose the Riemann tensor in its time and space projections (in each of its indices) with respect to a unit time-like vector U^α , $R^{\alpha\beta\gamma\delta} = (\top^\alpha_\rho + h^\alpha_\rho) \dots (\top^\delta_\sigma + h^\delta_\sigma) R^{\rho\dots\sigma}$, cf. Eq. (3); we obtain

$$\begin{aligned} R^{\alpha\beta}_{\gamma\delta} &= 4\mathbb{E}^{[\alpha}_{[\gamma} U_{\delta]} U^{\beta]} + 2 \left\{ \epsilon^{\mu\chi}_{\gamma\delta} U_\chi \mathbb{H}^{[\beta}_{\mu} U^{\alpha]} + \epsilon^{\mu\alpha\beta\chi} U_\chi \mathbb{H}_{\mu[\delta} U_{\gamma]} \right\} \\ &\quad + \epsilon^{\alpha\beta\phi\psi} U_\psi \epsilon^{\mu\nu}_{\gamma\delta} U_\nu \mathbb{F}_{\phi\mu}. \end{aligned} \quad (15)$$

where we made use of the identity (4) to project spatially an antisymmetric pair of indices, noting that $R_{\alpha\beta\gamma\delta}$ can be regarded as a double 2-form. This equation tells us that the Riemann tensor decomposes, with respect to U^α , in three *spatial* tensors: the gravitoelectric tidal tensor $\mathbb{E}_{\alpha\beta}$, the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}$, plus a third tensor

$$\mathbb{F}_{\alpha\beta} \equiv \star R \star_{\alpha\gamma\beta\delta} U^\gamma U^\delta = \epsilon^{\mu\nu}_{\alpha\gamma} \epsilon^{\lambda\tau}_{\beta\delta} R_{\mu\nu\lambda\tau} U^\gamma U^\delta$$

introduced by Bel² [48], which encodes the purely spatial curvature with respect to U^α , and has no electromagnetic analogue. In order to obtain Eq. (15), we made use of the symmetries $R^{\alpha\beta\gamma\delta} = R^{[\alpha\beta][\gamma\delta]}$, and in the case of the terms involving $\mathbb{H}_{\alpha\beta}$ (and *only for these terms*) we also assumed the pair exchange symmetry $R^{\alpha\beta\gamma\delta} = R^{\gamma\delta\alpha\beta}$. $\mathbb{E}_{\alpha\beta}$ and $\mathbb{F}_{\alpha\beta}$ are symmetric and therefore have 6 independent components each; $\mathbb{H}_{\alpha\beta}$ has 8, together encoding the 20 independent components of the Riemann tensor.

In what follows we will need also the Hodge dual, in the first two indices, of the decomposition (15):

$$\begin{aligned} \star R^{\alpha\beta}_{\gamma\delta} &= 2\epsilon^{\alpha\beta}_{\lambda\tau} \mathbb{E}^\lambda_{[\gamma} U_{\delta]} U^\tau + 4U^{[\alpha} \mathbb{H}^{\beta]}_{[\delta} U_{\gamma]} + \epsilon^{\alpha\beta}_{\lambda\tau} \epsilon^{\mu\nu}_{\gamma\delta} U_\nu \mathbb{H}^\tau_\mu U^\lambda \\ &\quad - 2U^{[\alpha} \mathbb{F}^{\beta]}_{\mu} \epsilon^{\mu\nu}_{\gamma\delta} U_\nu. \end{aligned} \quad (16)$$

The Ricci tensor $R^\beta_\delta = R^{\alpha\beta}_{\alpha\delta}$ and the tensor $\star R^\mu_{\alpha\mu\beta}$ follow as:

$$R^\beta_\delta = -\epsilon^{\alpha\beta\mu\nu} \mathbb{H}_{\mu\alpha} U_\delta U_\nu - \epsilon_{\alpha\delta\mu\nu} \mathbb{H}^{\mu\alpha} U^\beta U^\nu - \mathbb{F}^\beta_\delta - \mathbb{E}^\beta_\delta + \mathbb{E}^\sigma_\sigma U^\beta U_\delta + \mathbb{F}^\sigma_\sigma h^\beta_\delta, \quad (17)$$

$$\star R^{\alpha\beta}_{\alpha\delta} = \epsilon^{\alpha\beta}_{\lambda\tau} \mathbb{E}^\lambda_\alpha U_\delta U^\tau - \delta^\beta_\delta \mathbb{H}^\alpha_\alpha + U^\beta \mathbb{F}^\alpha_\mu \epsilon^{\mu\nu}_{\alpha\delta} U_\nu. \quad (18)$$

Substituting (17) in (14a), and (18) in (14b) we obtain Einstein's equations and the algebraic Bianchi identities in terms of the tensors $\mathbb{E}_{\alpha\beta}$, $\mathbb{H}_{\alpha\beta}$, $\mathbb{F}_{\alpha\beta}$. Now let us make the time-space splitting of these equations. Eq. (14a) is symmetric, hence it only has 3 non-trivial projections: time-time, time-space, and space-space. The time-time projection yields

$$\mathbb{E}^\alpha_\alpha = 4\pi (2\rho_m + T^\alpha_\alpha) \quad (19)$$

where $\rho_m \equiv T^{\alpha\beta} U_\beta U_\alpha$ denotes the mass-energy density as measured by an observer of 4-velocity U^α . Contraction of (17) and (14a) with the time-space projector $\top^\theta_\beta \epsilon^\delta_{\sigma\tau\gamma} U^\gamma$ yields:

$$\mathbb{H}_{[\sigma\tau]} = -4\pi \epsilon_{\lambda\sigma\tau\gamma} J^\lambda U^\gamma. \quad (20)$$

²The characterization of the Riemann tensor by these three spatial Rank 2 tensors is known as the ‘‘Bel decomposition’’; even though, for some reason, the explicit decomposition (15) has not, to the authors' knowledge, been written in the literature.

The space-space projection yields:

$$\mathbb{F}_\theta^\lambda + \mathbb{E}_\theta^\lambda - \mathbb{F}_\sigma^\sigma h_\theta^\lambda = 8\pi \left[h_\theta^\lambda \frac{1}{2} T_\alpha^\alpha - T_{\langle\theta}^{\langle\lambda} \right]. \quad (21)$$

where $T_{\langle\theta}^{\langle\lambda} \equiv h_\delta^\lambda h_\theta^\beta T_\beta^\delta$.

Eq. (14b), not exhibiting the tensor $\star R^{\gamma\alpha}_{\gamma\beta}$ a particular symmetry, seemingly splits in four parts: a time-time, time-space, space-time, and space-space projections. However only three of them are distinct. Substituting decomposition (18) in Eq. (14b), and taking the time-time, time-space, and space-time projections, yields, respectively:

$$\mathbb{H}_\alpha^\alpha = 0; \quad (a) \quad \mathbb{F}_{[\alpha\beta]} = 0; \quad (b) \quad \mathbb{E}_{[\alpha\beta]} = 0 \quad (c). \quad (22)$$

The space-space projection yields again \mathbb{H}_α^α , i.e., the same equation as the time-time projection.

These equations are summarized and contrasted with their electromagnetic counterparts in Table 1. Eqs. (1.3b)-(1.6b) are very similar in form to Maxwell Eqs. (1.3b)-(1.6b); they are their *physical* gravitational analogues, since both are the traces and antisymmetric parts of tensors $\{E_{\alpha\beta}, B_{\alpha\beta}\} \leftrightarrow \{\mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta}\}$ that we know, from equations (1.1) and (1.2), that play analogous physical roles in the two theories. Note this interesting aspect of the analogy: if one replaces, in Eqs. (9)-(12), the electromagnetic tidal tensors ($E_{\alpha\beta}$ and $B_{\alpha\beta}$) by the gravitational ones ($\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$), and the charges by masses (i.e., density ρ_c and current j^α of charge by density ρ and current J^α of mass) one *almost* obtains Eqs. (1.3b)-(1.6b), apart from a factor of 2 in the source term in (1.6b) and the difference in the source of Eq. (9), signaling that in gravity pressure and stresses contribute as sources. This happens because, since $\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ are spatial tensors, all the contractions with U^α present in Eqs. (10) and (12) vanish. That is, in the framework of the analogy based on tidal tensors, one starts with Maxwell equations, and by a simple application of the analogy one *almost* ends up with the *exact* temporal part of Einstein's equations.

The space-space part of Einstein equations, Eq. (1.7), has no electromagnetic analogue. It involves $\mathbb{F}_{\alpha\beta}$ and the space-space part of the energy momentum tensor $T^{\alpha\beta}$ as a source, none of them having an electromagnetic counterpart. This equation also has a fundamental difference³ with respect to the temporal equations (1.3b)-(1.6b) and their electromagnetic analogues: the latter are algebraic equations involving only the traces and antisymmetric parts of the tidal tensors, plus the source terms; they impose no condition on the symmetric parts. In electromagnetism, this is what allows the field to be dynamical, and waves to exist (their tidal tensors are described, in an inertial frame, by Eqs. (128)-(129) below); were there additional independent algebraic equations for the traceless symmetric part of the tidal tensors, and these fields would be fixed. But Eq. (21), by contrast, is an equation for the symmetric parts of the tensors $\mathbb{E}_{\alpha\beta}$ and $\mathbb{F}_{\alpha\beta}$. It can be split in two parts. Taking the trace, and using (1.3b), one obtains the source equation for $\mathbb{F}_{\alpha\beta}$:

$$\mathbb{F}_\sigma^\sigma = 8\pi\rho \quad (23)$$

substituting back in (1.7) we get:

$$\mathbb{F}_\beta^\alpha + \mathbb{E}_\beta^\alpha = 8\pi \left[h_\beta^\alpha \left(\frac{1}{2} T_\gamma^\gamma + \rho \right) - T_{\langle\beta}^{\langle\alpha} \right]. \quad (24)$$

³We thank João Penedones for drawing our attention to this point.

This equation tells us that the tensor \mathbb{F}_β^α is not an extra (comparing with electrodynamics) *independent* object; given the sources and the gravitoelectric tidal tensor $\mathbb{E}_{\alpha\beta}$, $\mathbb{F}_{\alpha\beta}$ is completely determined by (24).

A remarkable thing occurs however in vacuum ($T^{\alpha\beta} = 0$, $j^\alpha = 0$). The Riemann tensor becomes the Weyl tensor: $R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}$; due to the self duality property of the latter: $C_{\alpha\beta\gamma\delta} = -\star C \star_{\alpha\beta\gamma\delta}$, it follows $\mathbb{F}_{\alpha\beta} = -\mathbb{E}_{\alpha\beta}$, and therefore Eq. (1.7) *identically vanishes*. This means that in this case, the full Einstein equations and the algebraic Bianchi identity reduce to equations for the traces and antisymmetric parts of the the tidal tensors, just like the Maxwell equations. This is summarized in Table 2.

Table 2: In vacuum, the full Einstein equations, plus the algebraic Bianchi Identity, reduce to equations for the traces and antisymmetric parts of the tidal tensors, just like Maxwell’s equations.

Vacuum			
Maxwell’s Equations		Einstein Eqs. + Bianchi Id.	
$E^\alpha_\alpha = 0$	(2.1a)	$\mathbb{E}^\alpha_\alpha = 0$	(2.1b)
$E_{[\alpha\beta]} = U_{[\alpha} E_{\beta]\gamma} U^\gamma + \frac{1}{2} \epsilon_{\alpha\beta\mu\sigma} U^\sigma B^{\mu\gamma} U_\gamma$	(2.2a)	$\mathbb{E}_{[\alpha\beta]} = 0$	(2.2b)
$B^\alpha_\alpha = 0$	(2.3a)	$\mathbb{H}^\alpha_\alpha = 0$	(2.3b)
$B_{[\alpha\beta]} = U_{[\alpha} B_{\beta]\gamma} U^\gamma - \frac{1}{2} \epsilon_{\alpha\beta\mu\sigma} U^\sigma E^{\mu\gamma} U_\gamma$	(2.4a)	$\mathbb{H}_{[\alpha\beta]} = 0$	(2.4b)

These algebraic equations, as well as their general version (1.3)-(1.6) of Table 1, have the status of constraints for the tidal fields. They are especially suited to compare the tidal dynamics (i.e., Eqs. (1.1) and (1.2)) of the two interactions, which is discussed in the next section; such comparison allows us to learn some fundamental aspects of both interactions, which is exemplified, in the context of the dynamics of spinning test particles, in the companion paper [4]. But they do not tell us about the dynamics of the fields *themselves*. To obtain dynamical field equations, one possible route is to take one step back and express the tidal tensors in terms of gauge fields (such as the GEM “vector” fields \vec{G} , \vec{H} and the shear $K_{(\alpha;\beta)}$ of the 1+3 formalism of Sec. 3; the general expressions of Einstein equations in terms of these fields is given in Eqs. (7.9) and (7.3) of [19]); but it is also possible to write the equations for the dynamics of the tidal tensors (the *physical* fields); that is done not through Einstein equations (14), but through the differential Bianchi identity $R_{\sigma\tau[\mu\nu;\alpha]} = 0$, together with decomposition (15), and using (14) to substitute $R^{\alpha\beta}$ by the source terms. The resulting equations, for the case of vacuum (where $\{\mathcal{E}_{\alpha\beta}, \mathcal{H}_{\alpha\beta}\} = \{\mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta}\}$), are Eqs. (121)-(122) of Sec. 6 below. One may write as well dynamical equations for the electromagnetic tidal tensors, which for the case of vacuum, and an inertial frame, are Eqs. (128)-(129) of Sec. 6.1; however in the electromagnetic case the fundamental physical fields are the vectors E^α , B^α , whose covariant field equations are Eqs (54), (58), (60), (64) (the tidal field equations (128)-(129) follow trivially from these).

2.2 Gravity vs Electromagnetism

The tensor $\mathbb{F}_{\alpha\beta}$ — whereas Maxwell's equations can be fully expressed in terms of tidal tensors and sources, the same is only true, in general, for the temporal part of Einstein's equations. The Space-Space part, Eq. (21), involves the tensor $\mathbb{F}_{\alpha\beta}$, which has no electromagnetic analogue. $\mathbb{F}_{\alpha\beta}$ however is not an additional independent object, as it is completely determined via (21) given the sources and $\mathbb{E}_{\alpha\beta}$. In vacuum $\mathbb{F}_{\alpha\beta} = -\mathbb{E}_{\alpha\beta}$, and Einstein equations (plus the algebraic Bianchi Identity) reduce to equations for the traces and antisymmetric parts of the tidal tensors, analogous to their electromagnetic counterparts, cf. Table 2.

Sources — The source of the gravitational field is the rank two energy momentum tensor $T^{\alpha\beta}$, the source of the electromagnetic field is the current 4-vector j^α . Using the projectors (2) one can split $T^{\alpha\beta} = \rho U^\alpha U^\beta + 2h^\alpha_\mu J^{(\mu} U^{\beta)} + T^{\langle\alpha\rangle\langle\beta\rangle}$, and $j^\alpha = \rho_c U^\alpha + 2h^\alpha_\mu j^\mu$. Eqs. (1.3) show that the source of $E_{\alpha\beta}$ is ρ_c , and its gravitational analogue, as the source of $\mathbb{E}_{\alpha\beta}$, is $2\rho + T^\alpha_\alpha$ ($\rho + 3p$ for a perfect fluid). The magnetic/gravitomagnetic tidal tensors are analogously sourced by the charge/mass-energy currents $j^{\langle\mu\rangle}/J^{\langle\mu\rangle}$, as shown by Eqs. (1.6). Note that, in stationary (in the observer's frame) setups, $\star F_{\alpha\beta;\gamma} U^\gamma$ vanishes and equations (1.6a) and (1.6b) match up to a factor of 2, identifying $j^{\langle\mu\rangle} \leftrightarrow J^{\langle\mu\rangle}$. Eq. (23) shows that ρ is the source of $\mathbb{F}_{\alpha\beta}$. Eq. (1.7), sourced by the space-space part $T^{\langle\alpha\rangle\langle\beta\rangle}$, as well as the contribution T^α_α for (1.3b), manifest the well known fact that in gravity, by contrast with electromagnetism, pressure and stresses act as sources of the field.

Symmetries and time projections of Tidal Tensors — The gravitational and electromagnetic tidal tensors do not generically exhibit the same symmetries; also the former tidal tensors are spatial, whereas the latter have a time projection (with respect to the U^α measuring them), signaling fundamental differences between the two interactions. In the general case of fields that are time dependent in the observer's rest frame (that is the case of an intrinsically non-stationary field, or an observer moving in a stationary field), $E_{\alpha\beta}$ possesses an antisymmetric part, which is the covariant derivative of the Maxwell tensor along the observer's worldline; $\mathbb{E}_{\alpha\beta}$, by contrast, is always symmetric. As discussed above, $E_{\alpha\beta}$ is a covariant derivative of the electric field as measured in the MCRF; and Eq. (13a) is a covariant way of writing the Maxwell-Faraday equation $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$. Therefore, the statement encoded in the equation $\mathbb{E}_{[\alpha\beta]} = 0$, is that there is no *physical*, gravitational analogue to Faraday's law of induction (in the language of GEM vector fields of Sec. 3, we can say the the curl of the gravitoelectric field \vec{G} does not manifest itself in the tidal forces, unlike its electromagnetic counterpart. See Sec. 3.5 for explicit demonstration). To see a physical consequence, let δx^α in Eq. (1.1a) — the separation vector between a pair of particles with the same q/m and the same 4-velocity U^α — be spatial with respect to U^α ($\delta x^\alpha U_\alpha = 0$); and note that we can write the spatially projected antisymmetric part of $E_{\mu\nu}$ can be written in terms of the dual spatial vector α^μ : $E_{[\langle\mu\rangle\langle\nu\rangle]} = \epsilon_{\mu\nu\gamma\delta} \alpha^\gamma U^\delta$. Then the spatial components (1.1a) can be written as (using $E_{\langle\mu\rangle\langle\nu\rangle} = E_{(\langle\mu\rangle\langle\nu\rangle)} + E_{[\langle\mu\rangle\langle\nu\rangle]}$):

$$\frac{D^2 \delta x_{\langle\mu\rangle}}{d\tau^2} = \frac{q}{m} \left[E_{(\langle\mu\rangle\langle\nu\rangle)} \delta x^\nu + \epsilon_{\mu\nu\gamma\delta} \alpha^\gamma U^\delta \delta x^\nu \right] \Leftrightarrow \frac{D^2 \delta \vec{x}}{d\tau^2} = \frac{q}{m} \left[\overleftrightarrow{E} \cdot \delta \vec{x} + \delta \vec{x} \times \vec{\alpha} \right] \quad (25)$$

the second equation holding in the frame $U^i = 0$, and we used the dyadic notation \overleftrightarrow{E} of e.g. [105]. From the form of the second equation we see that $q\vec{\alpha}/m$ is *minus* an angular acceleration. Using relation (4), we see that $\alpha^\mu = -B^\mu_\beta U^\beta$; and in an inertial frame $\vec{\alpha} = \partial \vec{B}/\partial t = -\nabla \times \vec{E}$. In the

gravitational case, since $\mathbb{E}_{\mu\nu} = \mathbb{E}_{(\mu\nu)} = \mathbb{E}_{\langle\mu\rangle\langle\nu\rangle}$, we have

$$\frac{D^2\delta x_{\langle\mu\rangle}}{d\tau^2} = \frac{D^2\delta x_\mu}{d\tau^2} = \mathbb{E}_{(\mu\nu)}\delta x^\nu \quad \Leftrightarrow \quad \frac{D^2\delta\vec{x}}{d\tau^2} = \overleftrightarrow{\mathbb{E}} \cdot \delta\vec{x} \quad (26)$$

That is, given a set of neighboring charged test particles, the electromagnetic field “shears” them via $E_{(\mu\nu)}$, and induces an accelerated rotation via the laws of electromagnetic induction encoded in $E_{[\mu\nu]}$. The gravitational field, by contrast, only shears⁴ the particles, since $\mathbb{E}_{[\mu\nu]} = 0$.

Further physical evidence for the absence of a *physical* gravitational analogue for Faraday’s law of induction is given in the companion paper [4]: consider a spinning spherical charged body in an electromagnetic field; and choose the MCRF; if the magnetic field is not constant in this frame, by virtue of equation $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$ a torque will in general be exerted on the body by the induced electric field, changing its angular momentum and kinetic energy of rotation. By contrast, *no torque* is exerted in a spinning “spherical” body (i.e., a particle whose multipole moments in a local orthonormal frame match the ones of a spherical body in flat spacetime) placed in an *arbitrary* gravitational field; its angular momentum and kinetic energy of rotation are *constant*.

There is also an antisymmetric contribution $\star F_{\alpha\beta;\gamma}U^\gamma$ to $B_{\alpha\beta}$; in vacuum, Eq. (1.6a) is a covariant form of $\nabla \times \vec{B} = \partial\vec{E}/\partial t$; hence the fact that, in vacuum, $\mathbb{H}_{[\alpha\beta]} = 0$, means that there is no gravitational analogue to the antisymmetric part $B_{[\alpha\beta]}$ (i.e., the curl of \vec{B}) induced by the time varying field \vec{E} . Some physical consequences of this fact are explored in [4]: Eq. (1.6a) implies, via (1.2a), that whenever a magnetic dipole moves in a non-homogeneous field, it measures a non vanishing $B_{[\alpha\beta]}$ (thus also $B_{\alpha\beta} \neq 0$), and therefore (except for very special orientations of the dipole moment μ^α) a force will be exerted on it; in the gravitational case, by contrast, the gravitational force on a gyroscope is not constrained to be non-vanishing when it moves in a non-homogeneous field; it is found that it may actually move along geodesics, as is the case of radial motion in Schwarzschild spacetime⁵, or circular geodesics in Kerr-dS spacetime.

The spatial character of the gravitational tidal tensors, contrasting with their electromagnetic counterparts, is another difference in tensorial structure related to the laws of electromagnetic induction: as can be seen from Eqs. (10) and (12), the antisymmetric parts of the $E_{\alpha\beta}$ and $B_{\alpha\beta}$ (in vacuum, for the latter) consist of time projections of these tidal tensors. Physically, these time projections are manifest for instance in the fact that the electromagnetic force on a magnetic dipole has a non-vanishing projection along the particle’s 4-velocity U^α , which is the rate of work done on it by the induced electric field [1, 4], and is reflected in a variation of the particle’s proper mass. The projection, along U^α , of the gravitational force (1.2b), by its turn, vanishes, and the gyroscope proper mass is constant.

⁴If the two particles, instead of being free (i.e., moving along geodesics), were connected by a “rigid” rod, the symmetric part of the electric tidal tensors would also, in general, torque the rod; hence in such system, we would have a rotation, even in the gravitational case, see [17] pp. 154-155. The same for a quasi-rigid extended body; yet, even in this case the effect due to the symmetric parts are very different from the ones arising from electromagnetic induction: firstly the former do not require the fields to vary along the particle’s worldline, they exist even if the body is at rest in a stationary field; secondly, they vanish if the body is spherical, which does not happen with the torque generated by the induced electric field, see [4].

⁵This example is particularly interesting in this discussion. In the electromagnetic analogous problem: a magnetic dipole in (initially) radial motion in the Coulomb field of a point charge suffers a force; that force, as shown in [4], comes entirely from the antisymmetric part of the magnetic tidal tensor: $B_{\alpha\beta} = B_{[\alpha\beta]}$; it is thus a natural realization of the arguments above that $\mathbb{H}_{\alpha\beta} = 0$ in the analogous gravitational problem.

2.3 The analogy for differential precession

Eqs. (1.2) of Table 1 give the tensors $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ a physical interpretation as the tensors which, when contracted with a magnetic/gravitomagnetic dipole vector, yield the force exerted on magnetic dipoles/gyroscopes. We will now show that these tensors can also be interpreted as tensors of “relative”, or “differential”, precession for these test particles; i.e., tensors that, when contracted with a separation vector δx^β , yield the angular velocity of precession of a spinning particle (possessing magnetic moment) at given point \mathcal{P}_2 relative to a system of axes anchored to spinning particles, with *the same 4-velocity*, at the infinitesimally close point \mathcal{P}_1 . Somewhat analogous to the electric tidal tensors $E_{\alpha\beta}$ and $\mathbb{E}_{\alpha\beta}$, which, when contracted with δx^β , yield the relative acceleration of two infinitesimally close test particles with the same 4-velocity.

For clarity we will treat the gravitational and electromagnetic interactions separately. We will start by the gravitational problem. Consider a system of Fermi coordinates (see e.g. [100, 101]) with origin along a worldline passing through the location \mathcal{P}_1 , and denote by \mathbf{e}_α its basis vectors. Assuming Mathisson-Pirani spin condition $S^{\alpha\beta}U_\beta = 0$, the spin vector of a gyroscope undergoes Fermi-Walker transport, see e.g. [4]. Hence, a gyroscope at \mathcal{P}_1 , momentarily at rest in this frame (i.e., $\mathbf{U} = \mathbf{e}_0$) does not precess relative to it, as by definition the axes \mathbf{e}_α are Fermi-Walker transported therein. However at \mathcal{P}_2 , a gyroscope (with the same 4-velocity U^α) will be seen to precess relative to this frame; the angular velocity of precession is *minus* the angular velocity of rotation of the basis vectors \mathbf{e}_i relative to Fermi-Walker transport at \mathcal{P}_2 . Let X^α be a vector fixed with respect to the basis vectors at \mathcal{P}_2 ; it has the covariant derivative:

$$\frac{DX^i}{d\tau} = \frac{dX^i}{d\tau} + \Gamma_{\beta\gamma}^i(\mathcal{P}_2)U^\beta X^\gamma = \Gamma_{0j}^i(\mathcal{P}_2)X^j = R_{jk0}^i X^j \delta x^k$$

where we used the expression for the connection $\Gamma_{0j}^i(\mathcal{P}_2) = R_{jk0}^i \delta x^k$, see e.g. [101]. From Eq. (4), we note that

$$R_{\langle\alpha\rangle\langle\beta\rangle\gamma\tau} = \epsilon^{\mu\nu}_{\sigma\delta} \epsilon_{\mu\alpha\beta\lambda} U^\lambda U_\nu R^{\sigma\delta}_{\gamma\tau} = \epsilon_{\mu\alpha\beta\lambda} U^\lambda U_\nu \star R^{\mu\nu}_{\gamma\tau}$$

which, in the Fermi frame \mathbf{e}_α at \mathcal{P}_1 (orthonormal, where $U^i = 0$) reads: $R_{ij\gamma\tau} = -\epsilon_{ijk} \star R^{k0}_{\gamma\tau}$. We thus have $\Gamma_{0j}^i(\mathcal{P}_2) = -\epsilon^i_{jk} \star R^{k0}_{\gamma\tau} \delta x^l = -\epsilon^i_{jk} \mathbb{H}^k_l \delta x^l$ and therefore

$$\frac{D\vec{X}}{d\tau} = \vec{X} \times \delta\vec{\Omega}_G; \quad \delta\Omega_G^i \equiv \mathbb{H}^i_l \delta x^l \quad (27)$$

Hence, a gyroscope at \mathcal{P}_2 precesses with angular velocity $\delta\vec{\Omega}_G$ relative to a system of axes fixed to guiding gyroscopes at \mathcal{P}_1 (provided that the gyroscopes have the *same* 4-velocity U^α). This result was first obtained in a recent work [102] through a different procedure; we believe the derivation above is more clear, and shows that one of the assumptions made in [102] to obtain $\delta\vec{\Omega}_G$ — that the gyroscopes at \mathcal{P}_1 and \mathcal{P}_2 have the same acceleration — is not necessary, they only need to be momentarily comoving (i.e., have the same 4-velocity) in order for (27) to hold. The differential dragging effect, in terms of the eigenvalues of $\mathbb{H}_{\alpha\beta}$ and its associated field lines, and its visualization in different spacetimes, are discussed in detail therein.

Let us turn now to the analogous electromagnetic problem. Consider, in flat spacetime, a triad of orthogonal magnetic dipoles with 4-velocity U^α at \mathcal{P}_1 ; let $(\mathbf{e}_{\text{dip}})_{\hat{a}}$ be an orthonormal tetrad at \mathcal{P}_1 such that $(\mathbf{e}_{\text{dip}})_{\hat{0}} = \mathbf{U}$ and the spatial triad $(\mathbf{e}_{\text{dip}})_{\hat{i}}$ are axes attached to the dipoles. This will

be the reference system. If Mathisson-Pirani condition holds, the spin vector of a magnetic dipole (magnetic moment $\mu^\alpha = \sigma S^\alpha$, $\sigma \equiv$ gyromagnetic ratio) evolves as (e.g. [4]):

$$\frac{DS^\mu}{d\tau} = S_\nu a^\nu U^\mu + \sigma \epsilon^\mu_{\alpha\beta\nu} U^\nu S^\alpha B^\beta(\mathcal{P}_1) \quad (28)$$

whose space part reads:

$$\frac{D\vec{S}}{d\tau} = \vec{S} \times \vec{\Omega}(\mathcal{P}_1); \quad \vec{\Omega}(\mathcal{P}_1) \equiv \sigma \vec{B}(\mathcal{P}_1)$$

i.e., the magnetic dipoles (and spatial triad $(\mathbf{e}_{\text{dip}})_i$ attached to them) precess relative to Fermi-Walker transported axes with an angular velocity $-\vec{\Omega}(\mathcal{P}_1)$. Consider now a test magnetic dipole at \mathcal{P}_2 , momentarily moving with the same 4-velocity U^α . It will likewise precesses with an angular velocity $-\vec{\Omega}(\mathcal{P}_2) = -\sigma \vec{B}(\mathcal{P}_2)$ relative to Fermi-Walker transported axes at \mathcal{P}_2 . Now consider the global inertial frame (since we are in flat spacetime) *momentarily* comoving with the two sets of dipoles; the Fermi-Walker transport law for a spatial vector X^α ($U^\alpha X_\alpha = 0$), is $D\vec{X}/d\tau = 0$, which in the inertial frame reads $d\vec{X}/d\tau = 0$. That is, the spatial triads of Fermi-Walker transported tetrads *do not rotate* relative to a momentarily comoving inertial frame. And that effectively means that, in flat spacetime, the spatial axes of two Fermi-Walker transported tetrads along two distinct curves do not rotate one with respect to another provided that they are momentarily comoving (regardless of the acceleration of each of them). Hence, since the points \mathcal{P}_2 and \mathcal{P}_1 are taken to be momentarily at rest, the magnetic dipole at position \mathcal{P}_2 precesses with respect to the systems of Fermi Walker transported axes at \mathcal{P}_2 or \mathcal{P}_1 with the same frequency $-\vec{\Omega}(\mathcal{P}_2)$, and therefore with frequency $\delta\Omega_{\text{EM}} = \vec{\Omega}(\mathcal{P}_1) - \vec{\Omega}(\mathcal{P}_2)$ relative to the triad $(\mathbf{e}_{\text{dip}})_i$ attached to the reference dipoles at \mathcal{P}_1 . Again, since $U_1^\alpha = U_2^\alpha = U^\alpha$, it follows that

$$B^\alpha(\mathcal{P}_2) = B^\alpha(\mathcal{P}_1) + B^\alpha_{\gamma}(\mathcal{P}_1)\delta x^\gamma U^\beta + \mathcal{O}(\delta x^2)$$

where we used a Taylor expansion of $B^\alpha = F^\alpha_{\beta} U^\beta$ around \mathcal{P}_1 , and noted that $\star F^\alpha_{\beta,\gamma} \delta x^\gamma U^\beta = B^\alpha_{\gamma} \delta x^\gamma$. We obtain thus the relative precession rate

$$\delta\Omega_{\text{EM}}^i = \Omega^i(\mathcal{P}_1) - \Omega^i(\mathcal{P}_2) = -\sigma B^i_{\gamma} \delta x^\gamma \quad (29)$$

manifesting once more the physical analogy $\mathbb{H}_{\alpha\beta} \leftrightarrow B_{\alpha\beta}$, present also in the forces exerted on spinning particles, Eqs. (1.2), and in the field equations (1.6), (1.5) of Table 1.

3 Gravito-electromagnetic analogy based on inertial fields from the 1+3 splitting of spacetime

This approach has a different philosophy from the tidal tensor analogy of Sec. 2. Therein we aimed to compare physical, covariant forces of both theories; which was accomplished through the tidal forces. Herein the analogy drawn is between the electromagnetic fields E^α , B^α and spatial inertial fields G^α , H^α (i.e., fields of inertial forces, or “acceleration” fields), usually dubbed “gravitoelectromagnetic” (GEM) fields, that mimic E^α and B^α in gravitational dynamics. Inertial forces are fictitious forces, attached to a specific reference frame, and in this sense one can regard this analogy as a parallelism between physical forces from one theory, and reference frame effects from the other.

The GEM 3-vector fields are best known in the context of linearized theory for stationary spacetimes, e.g. [23, 24], where they are (somewhat naively) formulated as derivatives of the temporal parts of the linearized metric tensor (the GEM potentials, in analogy with the EM potentials). More general approaches are possible, if one observes that these are fields associated not to the local properties of a particular spacetime, but, as stated above, to the kinematical quantities of the reference frame. In particular, the GEM fields of the usual linearized approaches are but the acceleration and vorticity of the congruence of “static” observers $u^\alpha \simeq \delta_0^\alpha$ in the chosen background. Taking this perspective, the GEM fields may actually be cast in an *exact* form applying to arbitrary reference frames in arbitrary fields, through a general 1+3 splitting of spacetime. In this section we present such an exact and fully general formulation. The spacetime is split with respect to a congruence of observers of 4-velocity u^α , in a time direction parallel to u^α , and the 3-dimensional hyperplanes orthogonal to it, which has the interpretation of the rest space of the observers. Associating to each observer a tetrad $\mathbf{e}_{\hat{\alpha}}$ such that $\mathbf{e}_{\hat{0}} = \mathbf{u}$, and the spatial triad $\mathbf{e}_{\hat{i}}$ spans its local rest space, in the reference frame thereby defined the 4-dimensional physics is mapped to a (generalized, non-holonomic) curved 3-space where there “live” four spatial fields: G^α , Ω^α , ω^α and $K_{(\alpha\beta)}$, encoding the temporal part of the connections $(\Gamma_{\hat{0}\hat{0}}^{\hat{i}}, \Gamma_{\hat{0}\hat{j}}^{\hat{i}}, \Gamma_{\hat{j}\hat{0}}^{\hat{i}}, \Gamma_{\hat{j}\hat{j}}^{\hat{0}})$. G^α is the “gravitoelectric field” (*minus* acceleration of the observers), Ω^α and ω^α are, respectively, the rotation of the local tetrads relative to local Fermi-Walker transport and the vorticity of the congruence, which together form the gravitomagnetic field $H^\alpha = \Omega^\alpha + \omega^\alpha$. The rank 2 spatial tensor $K_{(\alpha\beta)}$ has no electromagnetic analogue, and consists of the shear and expansion of the observer congruence. The treatment herein is largely equivalent to the approach in [19, 20], only it is formulated herein in terms of an arbitrary reference frame, not necessarily the “congruence adapted” one (defined by setting $\Omega^\alpha = \omega^\alpha$, see below). The main difference (apart from the differences in the formalism) is that in this more general formulation the gravitomagnetic field H^α explicitly splits in the two *independent* parts ω^α and Ω^α ; this degree of generality is of use both in the companion paper [4], and in [63].

3.1 The reference frame

To an arbitrary observer of 4-velocity u^α at a given point x^α , one naturally associates an adapted frame (e.g. [19]), which is a tetrad $\mathbf{e}_{\hat{\alpha}}$ whose time axis is the observer’s 4-velocity $\mathbf{e}_{\hat{0}} = \mathbf{u}$ and whose spatial triad $\mathbf{e}_{\hat{i}}$ spans the local rest space of the observer. The latter is for now undefined up to an arbitrary rotation. The evolution of the tetrad along the observer’s worldline is generically described by the equation:

$$\nabla_{\mathbf{u}} \mathbf{e}_{\hat{\beta}} = \Omega_{\hat{\beta}}^{\hat{\alpha}} \mathbf{e}_{\hat{\alpha}}; \quad \Omega^{\alpha\beta} = 2u^{[\alpha} a^{\beta]} + \epsilon^{\alpha\beta}_{\mu\nu} \Omega^\mu u^\nu$$

where $\Omega^{\alpha\beta}$ is the (anti-symmetric) infinitesimal generator of Lorentz transformations, whose spatial part $\Omega_{\hat{i}\hat{j}} = \epsilon_{\hat{i}\hat{k}\hat{j}} \Omega^{\hat{k}}$ describes the arbitrary angular velocity $\vec{\Omega}$ of rotation of the spatial triad $\mathbf{e}_{\hat{i}}$ relative to a Fermi-Walker transported triad. Alternatively, from the definition of the connection coefficients, $\nabla_{\mathbf{e}_{\hat{\beta}}} \mathbf{e}_{\hat{\gamma}} = \Gamma_{\hat{\beta}\hat{\gamma}}^{\hat{\alpha}} \mathbf{e}_{\hat{\alpha}}$, we can think of the components of $\Omega^{\alpha\beta}$ as some of these coefficients:

$$\Omega_{\hat{0}}^{\hat{i}} = \Gamma_{\hat{0}\hat{0}}^{\hat{i}} = \Gamma_{\hat{0}\hat{i}}^{\hat{0}} = a^{\hat{i}}; \quad (30)$$

$$\Omega_{\hat{j}}^{\hat{i}} = \Gamma_{\hat{0}\hat{j}}^{\hat{i}} = \epsilon_{\hat{i}\hat{k}\hat{j}} \Omega^{\hat{k}}. \quad (31)$$

Unlike the situation in flat spacetime (and Lorentz coordinates), where one can take the tetrad adapted to a given observer as a global frame, in the general case such tetrad is a valid frame only locally, in an infinitesimal neighborhood of the observer. In order to define a reference frame over an extended region of spacetime, one needs a congruence of observers, that is, one needs to extend u^α to a field of unit timelike vectors. A connecting vector X^α between two neighboring observers in the congruence satisfies

$$[\mathbf{u}, \mathbf{X}] = \mathbf{0} \Leftrightarrow u^\beta \nabla_\beta X^\alpha - X^\beta \nabla_\beta u^\alpha = 0. \quad (32)$$

The evolution of the connecting vector along the worldline of an observer in the congruence is then given by the linear equation

$$\nabla_{\mathbf{u}} X^\alpha = \left(\nabla^\beta u^\alpha \right) X_\beta. \quad (33)$$

The component of the connecting vector orthogonal to the congruence,

$$Y^\alpha = X^\alpha + \left(u_\beta X^\beta \right) u^\alpha, \quad (34)$$

satisfies

$$\nabla_{\mathbf{u}} Y^\alpha = K^{\alpha\beta} Y_\beta + \left(a_\beta Y^\beta \right) u^\alpha, \quad (35)$$

where $K^{\alpha\beta}$ denotes the spatially projected covariant derivative of u^α the tensor

$$K^{\alpha\beta} \equiv (h^u)^\alpha{}_\lambda (h^u)^\beta{}_\tau u^{\lambda;\tau} = \nabla^\beta u^\alpha + a^\alpha u^\beta \quad (36)$$

The decomposition of this tensor into its trace, symmetric trace-free and anti-symmetric parts yields the expansion $\theta = \nabla_\alpha u^\alpha$, the shear

$$\sigma_{\alpha\beta} = K_{(\alpha\beta)} - \frac{1}{3}\theta g_{\alpha\beta} - \frac{1}{3}\theta u_\alpha u_\beta \quad (37)$$

and the vorticity

$$\omega_{\alpha\beta} = K_{[\alpha\beta]} \quad (38)$$

of the congruence. It is useful to introduce the vorticity vector

$$\omega^\alpha = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} u_{\gamma;\beta} u^\delta = -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} \omega_{\alpha\beta} u^\delta. \quad (39)$$

ω^α , according to definition above, yields *half* the curl of u^α ; this is in agreement with the convention in e.g. [14, 19], but differs by a minus sign from the definition in e.g. [32, 86]. Note however that for the vorticity tensor $\omega_{\alpha\beta}$ we are using the more general definition given in [32, 86], differing from a minus sign from the one in [19] (consequently, ω^α given by Eq. (39) is *minus* the dual of $\omega_{\alpha\beta}$). The nonvanishing tetrad components of $K_{\alpha\beta}$ are

$$K_{\hat{i}\hat{j}} = \sigma_{\hat{i}\hat{j}} + \frac{1}{3}\theta\delta_{\hat{i}\hat{j}} + \omega_{\hat{i}\hat{j}}. \quad (40)$$

These components determine the following connection coefficients:

$$K_{\hat{i}\hat{j}} = \nabla_{\hat{j}} u_{\hat{i}} = \Gamma_{\hat{j}\hat{i}}^{\hat{0}} = \Gamma_{\hat{j}\hat{0}}^{\hat{i}}. \quad (41)$$

The remaining temporal connection coefficients (other than the ones given in Eqs. (30)-(31), (41) above) are trivially zero:

$$\Gamma_{\hat{\alpha}\hat{0}}^{\hat{0}} = -\mathbf{e}_{\hat{0}} \cdot \nabla_{\mathbf{e}_{\hat{\alpha}}} \mathbf{e}_{\hat{0}} = -\frac{1}{2} \nabla_{\mathbf{e}_{\hat{\alpha}}} (\mathbf{e}_{\hat{0}} \cdot \mathbf{e}_{\hat{0}}) = 0.$$

Each observer of the congruence carries its own adapted tetrad, and to define the reference frame one must provide the law of evolution for the spatial triads orthogonal to u^α . A natural choice would be Fermi-Walker transport, $\vec{\Omega} = 0$ (the tetrad does not rotate relative to local guiding gyroscopes); another natural choice, of great usefulness in this framework, is to lock the rotation of the tetrads to the vorticity of the congruence, $\vec{\Omega} = \vec{\omega}$. This has been dubbed in the literature [19, 20] “co-rotating Fermi-Walker transport” (and the frame thereby defined is “the frame adapted to the congruence”, which is regarded as the most natural generalization of the non-relativistic concept of reference frame, see [38, 37]). This choice is more intuitive in the special case of a shear-free congruence, where, as we will show next, the axes of the frame thereby defined point towards *fixed* neighboring observers. Indeed, if X^α is a connecting vector between two neighboring observers of the congruence and Y^α is its component orthogonal to the congruence, we have

$$\begin{aligned} \nabla_{\mathbf{u}} Y^{\hat{i}} &= \dot{Y}^{\hat{i}} + \Gamma_{\hat{0}\hat{0}}^{\hat{i}} Y^{\hat{0}} + \Gamma_{\hat{0}\hat{j}}^{\hat{i}} X^{\hat{j}} \\ &= \dot{Y}^{\hat{i}} + \Omega_{\hat{j}}^{\hat{i}} X^{\hat{j}}. \end{aligned} \quad (42)$$

Since from (35) we have

$$\nabla_{\mathbf{u}} Y^{\hat{i}} = K_{\hat{j}}^{\hat{i}} Y^{\hat{j}}, \quad (43)$$

we conclude that

$$\dot{Y}_{\hat{i}} = \left(\sigma_{\hat{i}\hat{j}} + \frac{1}{3} \theta \delta_{\hat{i}\hat{j}} + \omega_{\hat{i}\hat{j}} - \Omega_{\hat{i}\hat{j}} \right) Y^{\hat{j}}. \quad (44)$$

This tells us that for a shear-free congruence ($\sigma_{\hat{i}\hat{j}} = 0$), if we lock the rotation $\vec{\Omega}$ of the tetrad to the vorticity $\vec{\omega}$ of the congruence, $\Omega_{\hat{i}\hat{j}} = \omega_{\hat{i}\hat{j}}$, the connecting vector’s direction is fixed on the tetrad (and if in addition $\theta = 0$, i.e., a rigid congruence, the connecting vectors have constant components on the tetrad). A familiar example is the rigidly rotating frame in flat spacetime; in the non-relativistic limit, the vorticity of the congruence formed by the rigidly rotating observers is constant, and equals the angular velocity of the frame; in this case, by choosing $\vec{\Omega} = \vec{\omega}$, one is demanding that the spatial triads $\mathbf{e}_{\hat{i}}$ carried by the observers co-rotate with the angular velocity of the congruence; hence it is clear that the axes $\mathbf{e}_{\hat{i}}$ always point to the same neighboring observers. For relativistic rotation, the vorticity $\vec{\omega}$ is not constant and no longer equals the angular velocity of the rotating observers; but the condition $\vec{\Omega} = \vec{\omega}$ still ensures that the tetrads are rigidly anchored to the observer congruence. Another example is the family of the so-called “static” observers in Kerr spacetime, which is very important in this context, because it is this construction which allows one to determine the rotation of the frame of the “distant stars” with respect to a local gyroscope, as explained in Sec. 3.3.

3.2 Geodesics — “gravitoelectromagnetic fields”

The equation of geodesics for a test particle of 4-velocity U^α , $\nabla_{\mathbf{U}} U^\alpha \equiv DU^\alpha/d\tau = 0$, reads, in the frame $e_{\hat{\alpha}}$:

$$\dot{U}^{\hat{i}} + \Gamma_{\hat{0}\hat{0}}^{\hat{i}} (U^{\hat{0}})^2 + \left(\Gamma_{\hat{0}\hat{j}}^{\hat{i}} + \Gamma_{\hat{j}\hat{0}}^{\hat{i}} \right) U^{\hat{0}} U^{\hat{j}} + \Gamma_{\hat{j}\hat{k}}^{\hat{i}} U^{\hat{k}} U^{\hat{j}} = 0$$

Substituting (30), (31) and (41), the spatial part becomes

$$\frac{\tilde{D}\vec{U}}{d\tau} = U^{\hat{0}} \left[U^{\hat{0}} \vec{G} + \vec{U} \times \vec{H} - \sigma^{\hat{i}}_{\hat{j}} U^{\hat{j}} \mathbf{e}_{\hat{i}} - \frac{1}{3} \theta \vec{U} \right] \quad (45)$$

where $\tilde{D}U^{\hat{i}}/d\tau = \dot{U}^{\hat{i}} + \Gamma^{\hat{i}}_{\hat{j}\hat{k}} U^{\hat{k}} U^{\hat{j}}$ denotes the tetrad components of the spatially projected covariant derivative, $\tilde{D}U^{\alpha}/d\tau \equiv (h^u)^{\alpha}_{\beta} DU^{\beta}/d\tau$, $\vec{G} = -\vec{a}$ is the “gravitoelectric field”, and $\vec{H} = \vec{\omega} + \vec{\Omega}$ is the “gravitomagnetic field”. These designations are due to the analogy with the roles that the electric and magnetic fields play in the electromagnetic Lorentz force, which in this notation reads

$$\frac{D\vec{U}}{d\tau} = \frac{q}{m_0} \left(U^0 \vec{E} + \vec{U} \times \vec{B} \right), \quad (46)$$

with $\vec{E} \equiv \vec{E}(u)$ and $\vec{B} \equiv \vec{B}(u)$ denoting the electric and magnetic fields as measured by the observers u^{α} . It is useful to write the GEM fields in a manifestly covariant form:

$$(G^u)^{\alpha} = -\nabla_{\mathbf{u}} u^{\alpha} \equiv -u^{\alpha}_{;\beta} u^{\beta}; \quad (H^u)^{\alpha} = \omega^{\alpha} + \Omega^{\alpha}, \quad (47)$$

The gravitomagnetic field $(H^u)^{\alpha}$ consists thus of two parts of different origins: the angular velocity Ω^{α} of rotation of the tetrads relative to Fermi-Walker transport (i.e., the local guiding gyroscopes), plus the vorticity ω^{α} of the congruence of observers u^{α} . If we lock the rotation of the tetrad to the vorticity of the congruence: $\Omega^{\alpha} = \omega^{\alpha}$, the gravitomagnetic field becomes simply twice the vorticity: $(H^u)^{\alpha} = 2\omega^{\alpha}$.

The last two terms of (45) have no electromagnetic counterpart; they correspond to the time derivative of the spatial metric. In fact, they involve only $K_{(\alpha\beta)}$, which is sometimes called the *second fundamental form* of the distribution of hyperplanes orthogonal to \mathbf{u} . If this distribution is integrable (that is, if there is no vorticity) then $K_{(\alpha\beta)}$ is just the extrinsic curvature of the time slices orthogonal to \mathbf{u} .

The quotient of the spacetime by the congruence is the “space of observers”. In the general case, where $K_{(\alpha\beta)} \neq 0$, there is no natural metric one can associate to this space, as the distance between observers, unlike the quotient, depends on the time slices. In the special case that the congruence is rigid, one naturally associates to this space the metric measuring the *constant* distance between neighboring observers, which is obtained by lifting tangent vectors on the quotient to tangent vectors orthogonal to the corresponding curve. On the other hand, if we have $K_{(\alpha\beta)} \neq 0$, but $\omega^{\alpha} = 0$, the congruence is hypersurface orthogonal, and it is natural to define a (time-dependent) 3-D metric on those hypersurfaces.

It is worth noting that Eq. (45) is the most general description of the geodesic motion possible in terms of electromagnetic-like fields (and kinematical quantities with no electromagnetic analogue); the corresponding results presented in the more popular linearized theory [24, 23] or Post-Newtonian [69, 47, 45] approaches are special cases of this equation (e.g. linearizing Eq. (45) above, one obtains Eq. (2.5) of [25]; further specializing to stationary fields, one obtains e.g. (6.1.26) of [23]).

3.2.1 Stationary fields — “Quasi-Maxwell” formalism

If one considers a stationary spacetime, *and* a frame where it is explicitly time-independent (i.e., a congruence of observers u^{α} tangent to a time-like Killing vector field, which necessarily means

that the congruence is rigid [118]), the last two terms of Eq. (45) vanish and the geodesic equation becomes formally similar to the Lorentz force (46):

$$\frac{\tilde{D}\vec{U}}{d\tau} = U^{\hat{0}} \left(U^{\hat{0}} \vec{G} + \vec{U} \times \vec{H} \right) \quad (48)$$

The line element of a *stationary spacetime* is generically described by:

$$ds^2 = -e^{2\Phi} (dt - \mathcal{A}_i dx^i)^2 + \gamma_{ij} dx^i dx^j \quad (49)$$

with Φ , $\vec{\mathcal{A}}$, γ_{ij} time-independent. γ_{ij} is a spatial metric, not flat in general, that measures the constant distance between stationary observers, as measured by the Einstein light signaling procedure [77]. This is the metric one can naturally associate with the quotient space in the case a rigid congruence, as discussed above. The GEM fields measured by the static observers (i.e. the observers of zero 3-velocity in the coordinate system of 49) are related with the metric potentials by [14]:

$$\vec{G} = -\tilde{\nabla}\Phi; \quad \vec{H} = e^{\Phi} \tilde{\nabla} \times \vec{\mathcal{A}}, \quad (50)$$

with $\tilde{\nabla}$ denoting the covariant differentiation operator *with respect to the spatial metric* γ_{ij} . The formulation (50) of GEM fields applying to stationary spacetimes is the most usual one; it was introduced in [77], and further worked out in e.g. [15, 14, 17, 93, 92], and is sometimes called the “Quasi-Maxwell formalism”.

3.3 Gyroscope precession

One of the main results of this approach is that within this formalism, the equation describing the evolution of the spin-vector of a gyroscope in a gravitational field (i.e., the Fermi-Walker transport law):

$$\frac{DS^\alpha}{d\tau} = S_\nu a^\nu U^\alpha, \quad (51)$$

takes, *when expressed in a local orthonormal tetrad comoving with the test particle*, a form exactly analogous to precession of a magnetic dipole under the action of a magnetic field. This analogy is more general than the one for the geodesics described above, it holds for arbitrary fields, and the observer does not have to be stationary (i.e. its worldline does not have to be tangent to a Killing vector); the only condition is to be comoving with the gyroscope.

Let U^α be the 4-velocity of the gyroscope; in a comoving orthonormal tetrad $e_{\hat{\alpha}}$, $U^{\hat{\alpha}} = \delta_{\hat{0}}^{\hat{\alpha}}$, and also $S^{\hat{0}} = 0$; therefore, Eq. (51) reduces in such frame to:

$$\frac{DS^{\hat{i}}}{d\tau} = 0 \Leftrightarrow \frac{dS^{\hat{i}}}{d\tau} = -\Gamma_{\hat{0}\hat{k}}^{\hat{i}} S^{\hat{k}} = \left(\vec{S} \times \vec{\Omega} \right)^{\hat{i}}$$

This result is somewhat obvious; note that we are just saying that, relative to this frame, gyroscopes, which are objects that “oppose” to changes in direction, and determine the the local “compass of inertia”, are seen to “precesses” with an angular velocity that is simply *minus* the angular velocity of rotation of the frame relative to Fermi-Walker transport. Now, for a congruence adapted frame, $\vec{\Omega} = \vec{\omega}$, this becomes:

$$\frac{d\vec{S}}{d\tau} = \frac{1}{2} \vec{S} \times \vec{H} \quad (52)$$

Table 3: The gravito-electromagnetic analogy based on inertial GEM fields.

Electromagnetism		Gravity	
Lorentz Force:		Geodesic Equation ($\vec{H} = \vec{\Omega} + \vec{\omega}$):	
$\frac{D\vec{U}}{d\tau} = \frac{q}{m_0} (U^0 \vec{E} + \vec{U} \times \vec{B})$	(3.1a)	$\frac{D\vec{U}}{d\tau} = U^0 \left[U^0 \vec{G} + \vec{U} \times \vec{H} - \sigma^i_j U^j \mathbf{e}_i - \frac{1}{3} \theta \vec{U} \right]$	(3.1b)
Precession of magnetic dipole:		Gyroscope “precession”:	
$\frac{D\vec{S}}{d\tau} = \vec{\mu} \times \vec{B}$	(3.2a)	$\frac{d\vec{S}}{d\tau} = \vec{S} \times \vec{\Omega}$	(3.2b)
Stationary fields, rigid, congruence adapted frame: $\vec{\Omega} = \vec{\omega} = \vec{H}/2$ (Quasi-Maxwell formalism)			
Force on magnetic dipole:		Force on gyroscope:	
$\frac{D\vec{P}}{d\tau} = \tilde{\nabla}(\vec{B} \cdot \vec{\mu}) - \frac{1}{2} \vec{\mu}(\tilde{\nabla} \cdot \vec{B}) - \frac{1}{2}(\vec{\mu} \cdot \vec{H}) \vec{E}$	(3.3a)	$\frac{D\vec{P}}{d\tau} = \frac{1}{2} \left[\tilde{\nabla}(\vec{H} \cdot \vec{S}) - \vec{S}(\tilde{\nabla} \cdot \vec{H}) - 2(\vec{S} \cdot \vec{H}) \vec{G} \right]$	(3.3b)
Maxwell Source Equations		Einstein Equations	
$F^{\alpha\beta}_{;\beta} = 4\pi J^\beta$		$R_{\mu\nu} = 8\pi (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\alpha_\alpha)$	
• Time Component:		• Time-Time Component:	
$\tilde{\nabla} \cdot \vec{E} = 4\pi \rho_c + \vec{H} \cdot \vec{B}$	(3.4a)	$\tilde{\nabla} \cdot \vec{G} = -4\pi(2\rho + T^\alpha_\alpha) + \vec{G}^2 + \frac{1}{2} \vec{H}^2$	(3.4b)
• Space Components:		• Time-Space Components:	
$\tilde{\nabla} \times \vec{B} = \vec{G} \times \vec{B} + 4\pi \vec{j}$	(3.5a)	$\tilde{\nabla} \times \vec{H} = 2\vec{G} \times \vec{H} - 16\pi \vec{J}$	(3.5b)
		• Space-Space Component:	
<i>No electromagnetic analogue</i>		$\tilde{\nabla}_i G_j - G_i G_j + \frac{1}{2} \vec{H}^2 \gamma_{ij} + \tilde{R}_{ij} = 8\pi (\frac{1}{2} \gamma_{ij} T^\alpha_\alpha + T_{ij})$	(3.6)
Bianchi Identity		Algebraic Bianchi Identity	
$\star F^{\alpha\beta}_{;\beta} = 0 \quad (\Leftrightarrow F_{[\alpha\beta;\gamma]} = 0)$		$\star R^{\gamma\alpha}_{\gamma\beta} = 0 \quad (\Leftrightarrow R_{[\alpha\beta\gamma]\delta} = 0)$	
• Time Component:		• Time-Time (or Space-Space) Component:	
$\tilde{\nabla} \cdot \vec{B} = -\vec{H} \cdot \vec{E}$	(3.7a)	$\tilde{\nabla} \cdot \vec{H} = -\vec{H} \cdot \vec{G}$	(3.7b)
• Space Components:		• Space-Time Components:	
$\tilde{\nabla} \times \vec{E} = \vec{G} \times \vec{E}$	(3.8a)	$\tilde{\nabla} \times \vec{G} = 0$	(3.8b)

Thus, the “precession” of a gyroscope is given, in terms of the gravitomagnetic field \vec{H} , by an expression identical (up to a factor of 2) to the precession of a magnetic dipole under the action of a magnetic field \vec{B} , cf. Eq. (28):

$$D\vec{S}/d\tau = \vec{\mu} \times \vec{B}. \quad (53)$$

This holds for arbitrary fields, and hence the result obtained for weak fields in [25] (that the analogy holds even if the fields are time dependent) is just a special case of this principle. However important differences should be noted: whereas in the electromagnetic case it is the same field \vec{B} that is at the origin of both the magnetic force $q(\vec{U} \times \vec{B})$ in Eq. (46) and the torque $\vec{\mu} \times \vec{B}$ on the magnetic dipole, in the case of the gravitomagnetic force $\vec{U} \times \vec{H}$ it has, in the general formulation, a

different origin from gyroscope “precession”, since the former arises not only from the rotation $\vec{\Omega}$ of the frame relative to a local Fermi-Walker transported tetrad, but also from the vorticity $\vec{\omega}$ of the congruence. Even in the case $\vec{\Omega} = \vec{\omega}$ there will be a relative factor of 2. In this sense, one can say that the Lense-Thirring effect detected in the LAGEOS satellite data [41] (and which will also be subject of experimental detection by the upcoming LARES mission [43]), measuring \vec{H} from test particle’s deflection, is of a different mathematical origin from the one which was under scrutiny by the Gravity Probe B mission [42], measuring $\vec{\Omega}$ from gyroscope precession, the two being made to match by measuring both effects relative to the “frame of the distant stars” (see below).

Another obvious difference is the presence of a covariant derivative in (53), and a simple derivative in (52), signaling that \vec{B} is a *physical* field, and \vec{H} a mere artifact of the reference frame (which can be anything, depending on the congruence of observers one chooses), that can be made to vanish by choosing a vorticity-free congruence. Note however that this does not mean that \vec{H} is necessarily meaningless; indeed it has no *local* physical significance, but it can tell us about frame dragging, which is a non-local physical effect. That is the case when one chooses the so-called “frame of the distant stars”, a notion that applies to asymptotically flat spacetimes. In stationary spacetimes, such frame is setup as follows: consider a rigid congruence of stationary observers such that at infinity it coincides with the asymptotic inertial rest frame of the source — the axes of the latter define the directions fixed relative to the distant stars. If the spacetime is also axisymmetric (such as the Kerr spacetime), this congruence *asymptotically* coincides with the zero angular momentum observers (ZAMOS, [117, 28, 17]). These observers are interpreted as being “at rest” with respect to the distant stars (and also at rest with respect to the asymptotic inertial frame of the source); since the congruence is rigid, it may be thought as a grid of points rigidly fixed to them. For this reason we dub them “static observers”⁶. This congruence fixes the time axis of the local tetrads of the frame. Now if we demand the rotation $\vec{\Omega}$ of the local tetrads (relative to Fermi-Walker transport) to equal the vorticity $\vec{\omega}$ of the congruence, Eq. (44) shows that the connecting vectors between different observers are constant in the tetrad; in other words, each local spatial triad e_i is locked to this grid, and therefore has directions fixed to the distant stars. Hence, despite having no *local* meaning, the gravitomagnetic field $\vec{H} = 2\vec{\Omega} = 2\vec{\omega}$ describes in this case a consequence of the frame dragging effect: the fact that a torque free gyroscope at finite distance from a rotating source precesses with respect to an inertial frame at infinity. This is a physical effect, that clearly distinguishes for instance the Kerr from the Schwarzschild spacetimes, but is non-local (i.e., it cannot be detected in any local measurement; only by locking to the distant stars by means of telescopes). It should be noted however that the relative precession of two neighboring (comoving) system of gyroscopes is locally measurable and encoded in the curvature tensor (more precisely, in the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}$, as discussed in Sec. 2.3).

⁶In the case of Kerr spacetime, these are the observers whose worldlines are tangent to the temporal Killing vector field $\xi = \partial/\partial t$, i.e., the observers of zero 3-velocity in Boyer Lindquist coordinates. This agrees with the convention in [117, 28]. We note however that the denomination “static observers” is employed in some literature (e.g. [103, 104]) with a different meaning, where it designates hypersurface orthogonal time-like Killing vector fields (which are rigid, *vorticity-free* congruences, existing only in *static spacetimes*).

3.4 Field equations

The Einstein field equations and the algebraic Bianchi Identity, Eqs. (14), can be generically written in this exact GEM formalism — i.e., in terms of \vec{G} , $\vec{\Omega}$, $\vec{\omega}$ and $K_{(\alpha\beta)}$. These equations are compared with the analogous electromagnetic situation: Maxwell's equations in an arbitrarily accelerated, rotating and shearing frame. The latter will be of use also in Sec. 6. We will also consider the special case of *stationary spacetimes* (and rigid, congruence adapted frames therein, $\vec{\Omega} = \vec{\omega} = \vec{H}/2$), where we recover the Quasi-Maxwell formalism of e.g. [14, 94, 77, 16, 93, 92]. In this case there is a closer similarity with the electromagnetic analogue — Maxwell's equations in an arbitrarily accelerated, rotating rigid frame (acceleration $\vec{a} = -\vec{G}$, vorticity $\vec{\omega} = \vec{\Omega} = \vec{H}/2$, cf. Sec. 3.1).

3.4.1 Maxwell equations for the electromagnetic fields measured by an arbitrary congruence of observers

Using decomposition (1), we write Maxwell's Eqs. (6) in terms of the electric and magnetic fields $(E^u)^\alpha = F^\alpha_\beta u^\beta$ and $(B^u)^\alpha = \star F^\alpha_\beta u^\beta$ measured by the congruence of observers of 4-velocity u^α . All the fields below are measured with respect to this congruence, so we may drop the superscripts: $(E^u)^\alpha \equiv E^\alpha$, $(B^u)^\alpha \equiv B^\alpha$. The time projection of Eq. (6a) with respect to u^α (see point 2 of Sec. 1.1) reads:

$$E^\beta_{;\beta} = 4\pi\rho_c + E^\alpha a_\alpha + 2\omega_\alpha B^\alpha \quad (54)$$

To write this equation in the tetrad components of the reference frame given in Sec. 3.1, we note that, for a given vector A^α :

$$A^\beta_{;\beta} = \left(A^{\hat{i}}_{;\hat{i}} + A^{\hat{i}}\Gamma^{\hat{j}}_{\hat{j}\hat{i}} \right) + A^{\hat{i}}\Gamma^{\hat{0}}_{\hat{0}\hat{i}} = \tilde{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{a}, \quad (55)$$

where $\tilde{\nabla}$ denotes the spatially projected covariant derivative of a tensor $A^{\beta_1 \dots \beta_n}$:

$$\tilde{\nabla}_\alpha A^{\beta_1 \dots \beta_n} \equiv (h^u)^\lambda{}_\alpha (h^u)^{\beta_1}{}_{\rho_1} \dots (h^u)^{\beta_n}{}_{\rho_n} \nabla_\lambda A^{\rho_1 \dots \rho_n}. \quad (56)$$

Mathematically, this covariant derivative is a connection in the vector bundle determined by the distribution of hyperplanes orthogonal to the congruence. *For the special case of a rigid congruence*, discussed in Sec. 3.2.1, it becomes a covariant derivative with respect to the natural spatial metric defined in the quotient space. Therefore, we have

$$\tilde{\nabla} \cdot \vec{E} = 4\pi\rho_c + 2\vec{\omega} \cdot \vec{B},$$

or, in the congruence adapted frame ($\vec{\omega} = \vec{\Omega} = \vec{H}/2$),

$$\tilde{\nabla} \cdot \vec{E} = 4\pi\rho_c + \vec{H} \cdot \vec{B}. \quad (57)$$

Analogously, for the time projection of (6b) we get

$$B^\beta_{;\beta} = B^\alpha a_\alpha - 2\omega^\mu E_\mu \quad (58)$$

which in the tetrad becomes

$$\tilde{\nabla} \cdot \vec{B} = -2\vec{\omega} \cdot \vec{E},$$

or

$$\tilde{\nabla} \cdot \vec{B} = -\vec{H} \cdot \vec{E} \quad (59)$$

in the congruence adapted frame. The space projection of Eq. (6a) reads:

$$\epsilon^{\alpha\gamma\beta} B_{\beta;\gamma} = \frac{D_F E^\alpha}{d\tau} - K^{(\alpha\beta)} E_\beta + \theta E^\alpha - \epsilon^\alpha_{\beta\gamma} \omega^\beta E^\gamma + \epsilon^\alpha_{\beta\gamma} B^\beta a^\gamma + 4\pi j^{(\alpha)} \quad (60)$$

where the index notation $\langle\mu\rangle$ stands for the spatially projected part of a vector: $V_{\langle\mu\rangle} \equiv h_\mu^\nu V_\nu$. Here we used $\epsilon^{\mu\beta\sigma} \equiv \epsilon^{\mu\beta\sigma\alpha} u_\alpha$, and the fact that the spatially projected covariant derivative of a *spatial* vector A^α equals its Fermi-Walker derivative:

$$(h^u)^\alpha_\beta \frac{DA^\beta}{d\tau} = \frac{D_F A^\alpha}{d\tau}.$$

$\tau \equiv \tau_u$ denotes herein the proper time along a worldline of tangent vector u^α . The tetrad components of (60) in the frame defined in Sec. 3.1 read:

$$(\tilde{\nabla} \times \vec{B})^{\hat{i}} = \frac{DE^{\hat{i}}}{d\tau} - K^{(\hat{i}\hat{j})} E_{\hat{j}} + \theta E^{\hat{i}} - (\vec{\omega} \times \vec{E})^{\hat{i}} + (\vec{G} \times \vec{B})^{\hat{i}} + 4\pi j^{\hat{i}}. \quad (61)$$

Using the connections (41), we note that the covariant derivative of a vector A^α can be written, in the tetrad, as:

$$\frac{DA^{\hat{i}}}{d\tau} = \frac{dA^{\hat{i}}}{d\tau} + \Gamma^{\hat{i}}_{\hat{0}\hat{j}} A^{\hat{j}} = \frac{dA^{\hat{i}}}{d\tau} + \Omega^{\hat{i}}_{\hat{j}} A^{\hat{j}}; \quad (62)$$

choosing the congruence adapted frame ($\Omega_{\hat{i}\hat{j}} = \omega_{\hat{i}\hat{j}}$), and substituting in (61), we obtain:

$$\tilde{\nabla} \times \vec{B} = \frac{d\vec{E}}{d\tau} + \vec{G} \times \vec{B} + 4\pi \vec{j} - K^{(\hat{i}\hat{j})} E_{\hat{j}} \vec{e}_{\hat{i}} + \theta \vec{E}; \quad (63)$$

in a rigid frame ($K^{(\hat{i}\hat{j})} = \theta = 0$) where the fields are time-independent ($d\vec{E}/d\tau = 0$), this yields Eq. (3.5a) of Table 3.

The space projection of (6b) is

$$\epsilon^{\alpha\gamma\beta} E_{\beta;\gamma} = -\frac{D_F B^\alpha}{d\tau} + K^{(\alpha\beta)} B_\beta - \theta B^\alpha + \epsilon^\alpha_{\beta\gamma} \omega^\beta B^\gamma + \epsilon^{\alpha\mu\sigma} E_\mu a_\sigma, \quad (64)$$

which analogously becomes, in the congruence adapted frame,

$$\tilde{\nabla} \times \vec{E} = -\frac{d\vec{B}}{d\tau} + \vec{G} \times \vec{E} + K^{(\hat{i}\hat{j})} B_{\hat{j}} \vec{e}_{\hat{i}} - \theta \vec{E}. \quad (65)$$

If the frame is rigid, and the fields time-independent, this yields Eq. (3.8a) of Table 3.

3.4.2 Einstein equations

We start by computing the tetrad components of the Riemann tensor in the frame of Sec. 3.1:

$$R_{\hat{0}\hat{i}\hat{0}\hat{j}} = -\tilde{\nabla}_{\hat{i}} G_{\hat{j}} + G_{\hat{i}} G_{\hat{j}} - \frac{d}{d\tau} K_{\hat{j}\hat{i}} + K_{\hat{l}\hat{i}} \Omega^{\hat{l}}_{\hat{j}} + \Omega^{\hat{l}}_{\hat{i}} K_{\hat{j}\hat{l}} - K^{\hat{l}}_{\hat{i}} K_{\hat{j}\hat{l}} \quad (66)$$

$$R_{\hat{j}\hat{k}\hat{i}\hat{0}} = \tilde{\nabla}_{\hat{j}} K_{\hat{i}\hat{k}} - \tilde{\nabla}_{\hat{k}} K_{\hat{i}\hat{j}} - 2G_{\hat{i}} \omega_{\hat{j}\hat{k}} \quad (67)$$

$$R_{\hat{i}\hat{j}\hat{k}\hat{l}} = \tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}} - K_{\hat{l}\hat{i}} K_{\hat{k}\hat{j}} + K_{\hat{l}\hat{j}} K_{\hat{k}\hat{i}} + 2\omega_{\hat{i}\hat{j}} \Omega_{\hat{k}\hat{l}} \quad (68)$$

In expressions above we kept $\vec{\Omega}$ independent of $\vec{\omega}$, so that they apply to an arbitrary orthonormal tetrad field. Here

$$\tilde{R}_{\hat{i}\hat{j}}^{\hat{l}} \equiv \Gamma_{\hat{j}\hat{k},\hat{i}}^{\hat{l}} - \Gamma_{\hat{i}\hat{k},\hat{j}}^{\hat{l}} + \Gamma_{\hat{i}\hat{m}}^{\hat{l}} \Gamma_{\hat{j}\hat{k}}^{\hat{m}} - \Gamma_{\hat{j}\hat{m}}^{\hat{l}} \Gamma_{\hat{i}\hat{k}}^{\hat{m}} - C_{\hat{i}\hat{j}}^{\hat{m}} \Gamma_{\hat{m}\hat{k}}^{\hat{l}} \quad (69)$$

denotes the Riemann tensor of the distribution of hyperplanes orthogonal to the congruence, that is, the curvature of the connection $\tilde{\nabla}$ on the vector bundle determined by the distribution. $\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}$ is anti-symmetric in the first and second pair of indices, but in general does not satisfy the algebraic Bianchi identities.

We shall now compute the tetrad components of the Ricci tensor, but now specializing to congruence adapted frames: $K_{[ij]} = \omega_{ij} = \Omega_{ij} = -\epsilon_{ijk}H^k/2$, so that the Ricci tensor comes in terms of the three GEM fields: \vec{G} , \vec{H} and $K_{(ij)}$. These read

$$R_{\hat{0}\hat{0}} = -\tilde{\nabla} \cdot \vec{G} + \vec{G}^2 + \frac{1}{2}\vec{H}^2 - \frac{d\theta}{d\tau} - K^{(\hat{i}\hat{j})}K_{(\hat{i}\hat{j})} \quad (70)$$

$$R_{\hat{0}\hat{i}} = \tilde{\nabla}^{\hat{j}}K_{(\hat{j}\hat{i})} - \theta_{,\hat{i}} + \frac{1}{2}(\tilde{\nabla} \times \vec{H})_{\hat{i}} - (\vec{G} \times \vec{H})_{\hat{i}} \quad (71)$$

$$R_{\hat{i}\hat{j}} = \tilde{R}_{\hat{i}\hat{j}} + \tilde{\nabla}_{\hat{i}}G_{\hat{j}} - G_{\hat{i}}G_{\hat{j}} + \frac{d}{d\tau}K_{(\hat{i}\hat{j})} + K_{(\hat{i}\hat{j})}\theta + \frac{1}{2}\left[\frac{d}{d\tau}H_{\hat{i}\hat{j}} + H_{\hat{i}\hat{j}}\theta + \vec{H}^2\delta_{\hat{i}\hat{j}} - H_{\hat{i}}H_{\hat{j}} + K_{(\hat{i}\hat{l})}H_{\hat{j}}^{\hat{l}} - H_{\hat{i}}^{\hat{l}}K_{(\hat{l}\hat{j})}\right], \quad (72)$$

where $H_{ij} = \epsilon_{ijk}H^k$ is the dual of \vec{H} , and $\tilde{R}_{\hat{i}\hat{j}} \equiv \tilde{R}_{\hat{i}\hat{l}\hat{j}}^{\hat{l}}$ is the Ricci tensor of the distribution of hyperplanes orthogonal to the congruence; this tensor is *not* symmetric in the general case of a congruence possessing both vorticity and shear. Using $T^{\hat{0}\hat{0}} = \rho$ and $T^{\hat{0}\hat{i}} = J^{\hat{i}}$, the time-time, time-space, and space-space components of the Einstein field equations with sources, Eq. (14a), read, respectively:

$$\tilde{\nabla} \cdot \vec{G} = -4\pi(2\rho + T_{\alpha}^{\alpha}) + \vec{G}^2 + \frac{1}{2}\vec{H}^2 - \frac{d\theta}{d\tau} - K^{(\hat{i}\hat{j})}K_{(\hat{i}\hat{j})} \quad (73)$$

$$\tilde{\nabla} \times \vec{H} = -16\pi\vec{J} + 2\vec{G} \times \vec{H} + 2\tilde{\nabla}\theta - 2\tilde{\nabla}_{\hat{j}}K^{(\hat{j}\hat{i})}\vec{e}_{\hat{i}} \quad (74)$$

$$8\pi\left(T_{\hat{i}\hat{j}} - \frac{1}{2}\delta_{\hat{i}\hat{j}}T_{\alpha}^{\alpha}\right) = \tilde{R}_{\hat{i}\hat{j}} + \tilde{\nabla}_{\hat{i}}G_{\hat{j}} - G_{\hat{i}}G_{\hat{j}} + \frac{d}{d\tau}K_{(\hat{i}\hat{j})} + K_{(\hat{i}\hat{j})}\theta + \frac{1}{2}\left[\frac{d}{d\tau}H_{\hat{i}\hat{j}} + H_{\hat{i}\hat{j}}\theta + \vec{H}^2\delta_{\hat{i}\hat{j}} - H_{\hat{i}}H_{\hat{j}} + K_{(\hat{i}\hat{l})}H_{\hat{j}}^{\hat{l}} - H_{\hat{i}}^{\hat{l}}K_{(\hat{l}\hat{j})}\right]. \quad (75)$$

Eqs. (73)-(74) are the gravitational analogues of the electromagnetic equations (57) and (63), respectively. Eq. (75) has no electromagnetic counterpart.

As for the algebraic Bianchi identities (14b), using (66)-(68), the time-time (equal to space-space), time-space and space-time components become, respectively:

$$\tilde{\nabla} \cdot \vec{H} = -\vec{G} \cdot \vec{H} \quad (76)$$

$$\tilde{\nabla} \times \vec{G} = -\frac{d\vec{H}}{d\tau} - \vec{H}\theta + H_{\hat{j}}K^{(\hat{j}\hat{i})}\vec{e}_{\hat{i}} \quad (77)$$

$$K_{(ij)}H^j = -\star\tilde{R}^j_{i} \quad (78)$$

Eqs. (76)-(77) are the gravitational analogues of the time and space projections of the electromagnetic Bianchi identities, Eqs. (59)-(65), respectively. Eq. (78) has no electromagnetic analogue.

This equations states that, when the observer congruence has both shear (including expansion) and vorticity, \tilde{R}_{ijkl} does not obey the algebraic Bianchi identities for a 3D curvature tensor. Note this remarkable aspect: all the terms in the Maxwell equations (57), (59) and (65) have a gravitational counterpart in (73), (76) and (77), respectively, substituting $\{\vec{E}, \vec{B}\} \rightarrow \{\vec{G}, \vec{H}\}$ and up to some numerical factors. As for (63), there are clear gravitational analogues in (74) to the terms $\vec{G} \times \vec{B}$ and the current $4\pi\vec{j}$, but not to the remaining terms. It should nevertheless be noted that, as shown in Sec. 5 below, in the Post-Newtonian regime (or in the “GEM limit” of linearized theory), the term $2\tilde{\nabla}\theta$ of (74) embodies a contribution analogous to the displacement current term $d\vec{E}/d\tau$ of (63). The gravitational equations contain, as one might expect, terms with no parallel in electromagnetism.

Most of the differing terms involve the total shear tensor $K_{(\hat{i}\hat{j})}$. The similarity thus get closer if we take the “Quasi-Maxwell” regime, i.e., stationary fields, and a frame adapted to a *rigid* congruence of stationary observers: $K_{ij} = K_{[ij]} = \omega_{ij}$. In this case the expressions above for the Riemann and Ricci tensors become the ones given in [14]. The gravitational field equations (14), i.e., (73)-(78), in this regime, are given in Table 3. Therein we drop the hats in the indices, for the following reason: as discussed in Sec. 3.2, in this regime there is a natural 3-D metric γ_{ij} on the quotient space (measuring the fixed distance between neighboring observers); we thus interpret the spatial fields \vec{G} and \vec{H} as vector fields on this 3-D Riemannian manifold. The spatially projected covariant derivative operator $\tilde{\nabla}$ becomes the covariant derivative of γ_{ij} (as $\Gamma_{jk}^i = {}^{(3)}\Gamma_{jk}^i$, i.e. the 4-D spatial connections equal the connection coefficients for γ_{ij}), and \tilde{R}_{ij} its Ricci tensor, which is symmetric (contrary to the general case). The equations in this “Quasi-Maxwell” regime exhibit a striking similarity with their electromagnetic counterparts, Eqs. (3.4a)-(3.8a) of Table 3, in spite of some natural differences that remain — numerical factors, the source and terms in (3.4b) with no electromagnetic counterpart. We note in particular that, by simply replacing $\{\vec{E}, \vec{B}\} \rightarrow \{\vec{G}, \vec{H}\}$ in (3.5a)-(3.8a), one obtains, up to some numerical factors, Eqs. (3.5b), (3.7b)-(3.8b). Of course, the electromagnetic terms involving products of GEM fields with EM fields, are mimicked in gravity by second order terms in the gravitational field. This is intrinsic to the non-linear nature of the gravitational field, and may be thought of as manifesting the fact that the gravitational field sources itself. Note in this context that the term $2\vec{G} \times \vec{H} \equiv -16\pi\vec{p}_G$ in Eq. (3.5b), sourcing the curl of the gravitomagnetic field, resembles the electromagnetic Poynting vector $\vec{p}_{EM} = \vec{E} \times \vec{B}/4\pi$; and the contribution $\vec{G}^2 + \vec{H}^2/2 \equiv -4\pi\rho_G$ in Eq. (3.5a), sourcing the divergence of the gravitoelectric field, resembles the electromagnetic energy density $\rho_{EM} = (E^2 + B^2)/8\pi$. For these reasons ρ_G and \vec{p}_G are dubbed in e.g. [92, 93, 94] gravitational “energy density” and “energy current density”, respectively. It is interesting to note that, in the asymptotic limit, \vec{p}_G corresponds to the time-space components of the Landau-Lifshitz [77] pseudo-tensor $t^{\mu\nu}$ [45]. One should however bear in mind that, by contrast with the electromagnetic counterparts, these quantities are artifacts of the reference frame, with no physical significance — at least from a *local* point of view (see related discussion in Sec. 6.1).

Finally, it is also interesting to compare Eqs. (73)-(78) with the tidal tensor version of the same equations, Eqs. (19)-(22).

3.5 Relation with tidal tensor formalism

The analogy based on the gravito-electromagnetic fields \vec{G} and \vec{H} is intrinsically different from the gravito-electromagnetic analogy based on tidal tensors introduced in [1]; the latter stems from tensor equations, whereas the former are fields of inertial forces, i.e., artifacts of the reference frame. A relationship between the two formalisms exists nevertheless, as in an arbitrary frame one can express the gravitational tidal tensors in terms of the GEM fields, using the expressions for the tetrad components of Riemann tensor Eqs. (66)-(67). This relationship is in many ways illuminating, as we shall see; it is one of the main results in this work, due the importance of using the two formalism together in practical applications, to be presented elsewhere [63]. Herein expressions are to be compared with the analogous electromagnetic situation, i.e., the electromagnetic tidal tensors computed from the fields as measured in an arbitrarily accelerating, rotating, and shearing frame (in flat or curved spacetime).

We start by the electromagnetic tidal tensors; from the definitions of $E_{\alpha\beta}$ and $B_{\alpha\beta}$ in Table 1, it follows that

$$E_{\alpha\gamma} = E_{\alpha;\gamma} - F_{\alpha\beta}U_{;\gamma}^{\beta} ; \quad B_{\alpha\gamma} = B_{\alpha;\gamma} - \star F_{\alpha\beta}U_{;\gamma}^{\beta} .$$

Using decompositions (1), and Eq. (62), we obtain the tetrad components ($E_{\hat{0}\hat{i}} = B_{\hat{0}\hat{i}} = 0$):

$$E_{\hat{i}\hat{j}} = \tilde{\nabla}_{\hat{j}}E_{\hat{i}} - \epsilon_{\hat{i}}^{\hat{l}\hat{m}}B_{\hat{m}}K_{\hat{l}\hat{j}} \quad (79)$$

$$B_{\hat{i}\hat{j}} = \tilde{\nabla}_{\hat{j}}B_{\hat{i}} + \epsilon_{\hat{i}}^{\hat{l}\hat{m}}E_{\hat{m}}K_{\hat{l}\hat{j}} \quad (80)$$

$$E_{\hat{i}\hat{0}} = \frac{dE_{\hat{i}}}{d\tau} + (\vec{\Omega} \times \vec{E})_{\hat{i}} + (\vec{G} \times \vec{B})_{\hat{i}} \quad (81)$$

$$B_{\hat{i}\hat{0}} = \frac{dB_{\hat{i}}}{d\tau} + (\vec{\Omega} \times \vec{B})_{\hat{i}} - (\vec{G} \times \vec{E})_{\hat{i}} \quad (82)$$

or, using $K_{ij} = \omega_{ij} + K_{(ij)}$, and choosing a congruence adapted frame ($\vec{\omega} = \vec{\Omega} = \vec{H}/2$),

$$E_{\hat{i}\hat{j}} = \tilde{\nabla}_{\hat{j}}E_{\hat{i}} - \frac{1}{2} \left[\vec{B} \cdot \vec{H} \delta_{\hat{i}\hat{j}} - B_{\hat{j}}H_{\hat{i}} \right] - \epsilon_{\hat{i}}^{\hat{l}\hat{m}}B_{\hat{m}}K_{(\hat{l}\hat{j})} \quad (83)$$

$$B_{\hat{i}\hat{j}} = \tilde{\nabla}_{\hat{j}}B_{\hat{i}} + \frac{1}{2} \left[\vec{E} \cdot \vec{H} \delta_{\hat{i}\hat{j}} - E_{\hat{j}}H_{\hat{i}} \right] + \epsilon_{\hat{i}}^{\hat{l}\hat{m}}E_{\hat{m}}K_{(\hat{l}\hat{j})} \quad (84)$$

$$E_{\hat{i}\hat{0}} = \frac{dE_{\hat{i}}}{d\tau} + \frac{1}{2}(\vec{H} \times \vec{E})_{\hat{i}} + (\vec{G} \times \vec{B})_{\hat{i}} \quad (85)$$

$$B_{\hat{i}\hat{0}} = \frac{dB_{\hat{i}}}{d\tau} + \frac{1}{2}(\vec{H} \times \vec{B})_{\hat{i}} - (\vec{G} \times \vec{E})_{\hat{i}} \quad (86)$$

Let us compute the gravitational tidal tensors. From the definitions of $\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ in Table 1, and using the tetrad components of the Riemann tensor, Eqs. (66)-(67), we obtain ($\mathbb{E}_{\hat{0}\hat{\alpha}} = \mathbb{E}_{\hat{\alpha}\hat{0}} = \mathbb{H}_{\hat{0}\hat{\alpha}} = \mathbb{H}_{\hat{\alpha}\hat{0}} = 0$):

$$\mathbb{E}_{\hat{i}\hat{j}} = -\tilde{\nabla}_{\hat{j}}G_{\hat{i}} + G_{\hat{i}}G_{\hat{j}} - \frac{d}{d\tau}K_{\hat{i}\hat{j}} + K_{\hat{l}\hat{j}}\Omega_{\hat{i}}^{\hat{l}} + \Omega_{\hat{j}}^{\hat{l}}K_{\hat{l}\hat{i}} - K_{\hat{j}}^{\hat{l}}K_{\hat{l}\hat{i}} \quad (87)$$

$$\mathbb{H}_{\hat{i}\hat{j}} = -\tilde{\nabla}_{\hat{j}}\omega_{\hat{i}} + \delta_{\hat{i}\hat{j}}\tilde{\nabla} \cdot \vec{\omega} + 2G_{\hat{j}}\omega_{\hat{i}} + \epsilon_{\hat{i}}^{\hat{l}\hat{m}}\tilde{\nabla}_{\hat{l}}K_{(\hat{j}\hat{m})} \quad (88)$$

For a congruence adapted frame these expressions become:

$$\begin{aligned} \mathbb{E}_{\hat{i}\hat{j}} &= -\tilde{\nabla}_{\hat{j}} G_{\hat{i}} + G_{\hat{i}} G_{\hat{j}} + \frac{1}{4} \left(\vec{H}^2 \gamma_{ij} - H_j H_i \right) + \frac{1}{2} \epsilon_{\hat{i}\hat{j}\hat{k}} \frac{dH^{\hat{k}}}{d\tau} + \epsilon_{\hat{j}\hat{m}} H^{\hat{m}} K_{(\hat{i}\hat{l})} \\ &\quad - \frac{d}{d\tau} K_{(\hat{i}\hat{j})} - \delta_{\hat{l}}^{\hat{m}} K_{(\hat{i}\hat{l})} K_{(\hat{m}\hat{j})} \end{aligned} \quad (89)$$

$$\mathbb{H}_{\hat{i}\hat{j}} = -\frac{1}{2} \left[\tilde{\nabla}_{\hat{j}} H_i + (\vec{G} \cdot \vec{H}) \gamma_{ij} - 2G_j H_i \right] + \epsilon_{\hat{i}}^{\hat{m}} \tilde{\nabla}_{\hat{l}} K_{(\hat{j}\hat{m})} \quad (90)$$

In (90) we substituted $\tilde{\nabla} \cdot H = \vec{G} \cdot \vec{H}$ using Eq. (77). Note the formal similarities with the electromagnetic analogues (83)-(84). All the terms present in E_{ij} and B_{ij} , except for the last term of the latter, have a correspondence in their gravitational counterparts \mathbb{E}_{ij} , \mathbb{H}_{ij} , substituting $\{\vec{E}, \vec{B}\} \rightarrow -\{\vec{G}, \vec{H}\}$ and correcting some factors of 2. However, the gravitational tidal tensors contain additional terms which (together with the differing numerical factors) encode the crucial differences in the tidal dynamics of the two interactions. The third and forth terms in (83), which have no electromagnetic analogue, have the role of canceling out the antisymmetric part of $\tilde{\nabla}_{\hat{j}} G_{\hat{i}}$, that is, canceling out the contribution of the curl of \vec{G} to the gravitoelectric tidal tensor, as can be seen from Eq. (77). In particular, note that the term $-dH^i/d\tau$, “inducing”, via Eq. (77), the curl of \vec{G} (which might lead one to predict gravitational induction effects in analogy with Faraday’s law of electromagnetism), is being subtracted in (89), meaning that the curl of \vec{G} does not translate in the *physical, covariant* forces. For instance, it does not induce rotation in a set of free neighboring particles (see Eq. (26) above and discussion therein), nor does it torque an extended rigid body, as shown in the companion paper [4].

There are some interesting special regimes where the relation between the tidal tensor and the inertial fields formalism becomes simpler. One is the “Quasi-Maxwell” regime, i.e., *stationary spacetimes*, and a *frame adapted to a rigid (i.e., shear and expansion-free) congruence of stationary observers*. The gravitational tidal tensors *as measured in such frame* can be expressed entirely in terms of the gravito-electric \vec{G} and gravitomagnetic \vec{H} fields; the non-vanishing components are:

$$\mathbb{E}_{ij} = -\tilde{\nabla}_j G_i + G_i G_j + \frac{1}{4} \left(\vec{H}^2 \gamma_{ij} - H_j H_i \right); \quad (91)$$

$$\mathbb{H}_{ij} = -\frac{1}{2} \left[\tilde{\nabla}_j H_i + (\vec{G} \cdot \vec{H}) \gamma_{ij} - 2G_j H_i \right]. \quad (92)$$

The hats in the indices of these expressions are dropped because, since in the Quasi-Maxwell regime there is a natural metric γ_{ij} associated to the quotient space (see Sec. 3.2), we express these tensors in terms of an arbitrary (possibly coordinate) basis in the quotient space (as we did in Sec. 3.2.1 and in Table 3), instead of tetrad components.

The non-vanishing components of the electromagnetic tidal tensors are, under the same conditions,

$$E_{ij} = \tilde{\nabla}_j E_i - \frac{1}{2} \left[\vec{B} \cdot \vec{H} \gamma_{ij} - B_j H_i \right] \quad (a) \quad E_{i0} = \frac{1}{2} (\vec{H} \times \vec{E})_i + (\vec{G} \times \vec{B})_i \quad (b) \quad (93)$$

$$B_{ij} = \tilde{\nabla}_j B_i + \frac{1}{2} \left[\vec{E} \cdot \vec{H} \gamma_{ij} - E_j H_i \right] \quad (a) \quad B_{i0} = \frac{1}{2} (\vec{H} \times \vec{B})_i - (\vec{G} \times \vec{E})_i \quad (b) \quad (94)$$

Thus again, even in the stationary regime, the electromagnetic tidal tensors have non-vanishing time components, unlike their gravitational counterparts. The spatial parts, however, are very

similar in form; note that replacing $\{\vec{E}, \vec{B}\} \rightarrow -\{\vec{G}, \vec{H}/2\}$ in (94), the time components vanish, and one *almost* obtains the space part (88), apart from the factor of 2 in the third term; and that a similar substitution in (93) *almost* leads to (91), apart from the term $G_i G_j$, which has no electromagnetic counterpart. The gravitational and electromagnetic tidal tensors are nevertheless very different, even in this regime; namely in their symmetries. E_{ij} is not symmetric, whereas \mathbb{E}_{ij} is (the second and third terms in (91) are obviously symmetric; and that the first one also is can be seen from Eq. (3.8b) of Table 3). As for the magnetic tidal tensors, note that, by virtue of Eq. (3.5b), the last term of (92) ensures that, in vacuum, the antisymmetric part $H_{[i;j]}$ (i.e., the curl of \vec{H}) is subtracted from $H_{i;j}$ in (88), thus keeping \mathbb{H}_{ij} symmetric, by contrast with B_{ij} . This can be seen explicitly by noting that *in vacuum* (92) can be put in the equivalent form:

$$\mathbb{H}_{ij} = -\frac{1}{2} \left[H_{i;j} - H_{[i;j]} + (\vec{G} \cdot \vec{H}) \gamma_{ij} - 2G_{(j} H_{i)} \right]$$

where we used $H_{[i;j]} = 2G_{[j} H_{i]}$, as follows from Eq. (3.5b).

Another interesting regime to consider is the weak field limit, where the non-linearities of the gravitational field are negligible, and compare with electromagnetism in inertial frames. From Eqs. (83)-(86), the non-vanishing components for the electromagnetic tidal tensors measured by observers at rest in an inertial frame are:

$$E_{ij} = E_{i,j}; \quad E_{i0} = \frac{dE_i}{d\tau}; \quad B_{ij} = B_{i,j} \quad B_{i0} = \frac{dB_i}{d\tau},$$

i.e., they reduce to ordinary derivatives of the electric and magnetic fields. The linearized gravitational tidal tensors are, from Eqs. (89)-(90):

$$\mathbb{E}_{ij} \approx -G_{i,j} + \frac{1}{2} \epsilon_{ijk} \frac{dH^k}{d\tau} - \frac{d}{d\tau} K_{(ij)}; \quad (a) \quad \mathbb{H}_{ij} \approx -\frac{1}{2} H_{i,j} + \epsilon_i^{lm} K_{(jm),l}. \quad (b) \quad (95)$$

Thus, even in the linear regime, the gravitational tidal tensors cannot, in general, be regarded as derivatives of the gravitoelectromagnetic fields \vec{G} and \vec{H} . Noting, from Eq. (107) below, that $K_{(ij)}$ is the time derivative of the spatial metric, we see that *only if the fields are time independent* in the chosen frame, do we have $\mathbb{E}_{ij} \approx -G_{i,j}$, $\mathbb{H}_{ij} \approx -\frac{1}{2} H_{i,j}$.

3.6 Force on a gyroscope

In the framework of the 1+3 formalism, there is also an analogy [14] relating the gravitational force on a gyroscope and the electromagnetic force on a magnetic dipole. This is an analogy different from the one based on tidal tensors, and not as general. We start with equations (1.2) of Table 1, which tell us that the forces are determined by the magnetic/gravitomagnetic tidal tensors as *seen by the particle*. For the spatial part of the forces, only the space components of the tidal tensors, as measured in the particle's proper frame, contribute. Comparing Eqs. (84) and (90), which yield the tidal tensors in terms of the electromagnetic/gravitoelectromagnetic fields, we see that a close formal analogy is possible only when $K_{(\alpha\beta)} = 0$ in the chosen frame. Thus, a close analogy between the forces in this formalism can hold only when the particle is at rest with respect to a congruence for which $K_{(\alpha\beta)} = 0$; that is, a rigid congruence. The rigidity requirement can be satisfied only in special spacetimes [118]; it is ensured in the ‘‘Quasi-Maxwell’’ regime — that is, stationary spacetimes, and congruences tangent to time-like Killing vector fields therein.

Let us start by the electromagnetic problem — a magnetic dipole at rest in a rigid, but arbitrarily accelerating and rotating frame. Since the dipole is at rest in that frame, $\mu^\alpha = (0, \mu^i)$; hence the spatial part of the force is $F_{EM}^i = B^{ji}\mu_j$. Substituting (94a) in this expression yields the force exerted on the dipole, in terms of the electric and magnetic fields *as measured in its proper frame*:

$$\vec{F}_{EM} = \tilde{\nabla}(\vec{B} \cdot \vec{\mu}) + \frac{1}{2} \left[\vec{\mu}(\vec{E} \cdot \vec{H}) - (\vec{\mu} \cdot \vec{H})\vec{E} \right] \quad (96)$$

Using $\vec{H} \cdot \vec{E} = -\tilde{\nabla} \cdot \vec{B}$, cf. Eq. (3.7a), we can re-write this expression as

$$\vec{F}_{EM} = \tilde{\nabla}(\vec{B} \cdot \vec{\mu}) - \frac{1}{2} \left[\vec{\mu}(\tilde{\nabla} \cdot \vec{B}) + (\vec{\mu} \cdot \vec{H})\vec{E} \right] \quad (97)$$

Consider now the analogous electromagnetic situation: a gyroscope at rest (i.e., with zero 3-velocity, $U^i = 0$) with respect to stationary observers (arbitrarily accelerated and rotating) in a stationary gravitational field; from Eqs. (1.2b) and (88), the force exerted on it is given by:

$$\vec{F}_G = \frac{1}{2} \left[\tilde{\nabla}(\vec{H} \cdot \vec{S}) + \vec{S}(\vec{G} \cdot \vec{H}) - 2(\vec{S} \cdot \vec{H})\vec{G} \right] \quad (98)$$

From Eq. (3.7b) we have $\vec{G} \cdot \vec{H} = -\tilde{\nabla} \cdot \vec{H}$; substituting yields [14]:

$$\vec{F}_G = \frac{1}{2} \left[\tilde{\nabla}(\vec{H} \cdot \vec{S}) - \vec{S}(\tilde{\nabla} \cdot \vec{H}) - 2(\vec{S} \cdot \vec{H})\vec{G} \right] \quad (99)$$

Note that replacing $\{\vec{\mu}, \vec{E}, \vec{B}\} \rightarrow \{\vec{S}, \vec{G}, \vec{H}/2\}$ in Eq. (96), one *almost* obtains (98), except for a factor of 2 in the last term. The last term of (98)-(99), in this framework, can be interpreted as the “weight” of the dipole’s energy [14]. It plays, together with Eq. (3.5b), a crucial role in the dynamics, as it cancels out the contribution of the curl of \vec{H} to the force, ensuring that, in the tidal tensor form (1.2b), it is given by a contraction of S^α with a *symmetric* tensor $\mathbb{H}_{\alpha\beta}$. This is discussed in detail in Sec. 3.5. This contrasts with the electromagnetic case, where the curl of \vec{B} is manifest in $B_{\alpha\beta}$ (which has an antisymmetric part) and in the force. The expression (99) was first found in [14] (where it was compared to the force on a magnetic dipole as measured in the *inertial* frame *momentarily* comoving with it, case in which the last two terms of 97 vanish); herein we add expression (97), which is its electromagnetic counterpart for analogous conditions (the frame where the particle is at rest, which can be an accelerating and rotating), and shows that the analogy is even stronger.

4 “Ultra-stationary” spacetimes

Ultra-stationary spacetimes are stationary spacetimes admitting *rigid* geodesic time-like congruences. In the coordinate system adapted to such congruence, the metric is generically obtained by taking $\Phi = 0$ in Eq. (49), leading to:

$$ds^2 = - \left(dt - \mathcal{A}_i(x^k) dx^i \right)^2 + \gamma_{ij}(x^k) dx^i dx^j . \quad (100)$$

Examples of these spacetimes are the Som-Raychaudhuri [88], Van-Stockum interior solution [89], and Gödel [90] spacetimes; see [2] for their discussion in this context. This is an interesting class

of spacetimes in the context of GEM, due to the close *similarity* with electrodynamics, which was explored in an earlier work [1] by one of the authors: 1) they are exactly mapped [68, 1], via the Klein-Gordon equation, in curved 3-spaces with a gravitomagnetic field; 2) their gravitomagnetic tidal tensor is linear [1] (just like in the case of electromagnetism), and, up to a factor, matches the covariant derivative of the gravitomagnetic field. A link between these two properties was suggested therein⁷; however, the non-vanishing gravitoelectric tidal tensor (while no gravitoelectric part is present in the map) was a question left unanswered. Herein, putting together the knowledge from the tidal tensor with the inertial force formalisms (Secs. 2 and 3), we revisit these spacetimes and shed some new light on these issues.

Eqs. (50) yield the GEM fields corresponding to the frame adapted to the rigid geodesic congruence, $u^\alpha = \delta_0^\alpha$. They tell us that the gravito-electric field vanishes: $\vec{G} = 0$, which is consistent with the fact that no electric field arises in the mapping above; the gravitomagnetic field \vec{H} is *linear* in the metric potentials:

$$\vec{H} = \tilde{\nabla} \times \vec{\mathcal{A}}. \quad (101)$$

These properties can be interpreted as follows. The fact that $\vec{G} = 0$ means that the metric is written in a frame corresponding to a congruence of freely falling observers (as their acceleration $a^\alpha = -G^\alpha$ is zero); the very special property of these spacetimes (unlike the situation in general, e.g. the Kerr or Schwarzschild spacetimes) is that such congruence is rigid, i.e. has no shear, allowing the metric to be time independent in the coordinates associated to that frame. The gravitomagnetic field, on the other hand, does not vanish in this frame, which means in this context (since the frame is congruence adapted, see Sec. 3.1), that the congruence has vorticity. The equation of motion for a free particle in this frame, cf. Eq. (48), reduces to :

$$\frac{\tilde{D}\vec{U}}{d\tau} = U^0 \vec{U} \times \vec{H}, \quad (102)$$

similar to the equation of motion of a charged particle under the action of a magnetic field; and since \vec{H} is a linear function of the metric, the similarity with the electromagnetic analogue is indeed close.

Let us now examine the tidal effects. This type of spacetimes have a very special property: the gravitomagnetic tidal tensor measured by the observers $u^\alpha = \delta_0^\alpha$ is *linear* in the fields (and thus in the metric potentials), cf. Eq. (88), and, just like in the electromagnetic analogue, it is given by the covariant derivative of \vec{H} with respect to γ_{ij} :

$$\mathbb{H}_{ij} = -\frac{1}{2}H_{i;j} = -\tilde{\epsilon}^{lk}{}_i \mathcal{A}_{k;l;j}. \quad (103)$$

($\mathbb{H}_{0j} = \mathbb{H}_{00} = \mathbb{H}_{j0}$ for these observers) where semi-colons denote covariant derivatives with respect to γ_{ij} . This reinforces the similarity with electromagnetism. The gravitoelectric tidal tensor is,

⁷In an earlier work by one of the authors [1, 2] (to whom the *exact* GEM fields analogy of Sec. 3 was not yet known), it was suggested that the above mapping could be interpreted as arising from the similarity of magnetic tidal forces manifest in relations (103). It seems, however, to be much more related to the analogy based on GEM “vector” fields manifest in Eqs. (101) and (102). Even though the exact correspondence (103) reinforces in some sense the analogy, tidal forces do not seem to be the underlying principle behind the mapping, since: i) in it there is no electromagnetic counterpart to the non-vanishing gravitoelectric tidal tensor $\mathbb{E}_{\alpha\beta}$; ii) the Klein-Gordon Eq. $\square\Phi = m^2\Phi$ and the Hamiltonian in Sec. IV of [1] is for a (free) monopole particle, no tidal forces contribute. It is thus expected to reveal coordinate artifacts such as the fields \vec{G} , \vec{H} , and not physical tidal forces.

however, non-zero, as seen from Eq. (87):

$$\mathbb{E}_{ij} = \frac{1}{4} \left(\vec{H}^2 \gamma_{ij} - H_j H_i \right) , \quad (104)$$

even though $\vec{G} = 0$; and that should not be surprising, for the following reasons: i) it is always possible to make \vec{G} vanish by choosing freely falling observers (this is true in an arbitrary spacetime), but that does not eliminate the tidal effects, as they arise from the curvature tensor; ii) in the present case of ultrastationary spacetimes, $\mathbb{E}_{\alpha\beta}$ is actually a non-linear tensor in \vec{H} ; which merely reflects the fact that, except on very special circumstances, $\mathbb{E}_{\alpha\beta}$ (unlike its electromagnetic counterpart) cannot be thought as simply a covariant derivative of some vector field \vec{G} .

The tidal tensor (104) exhibits other interesting properties. It vanishes along the direction of the gravitomagnetic field H^α : if X^α is a vector spatial with respect to u^α (i.e., $X^\alpha u_\alpha = 0$) which is parallel to the gravitomagnetic field H^α then the tidal force $-\mathbb{E}^\alpha_\beta X^\beta$ measured by the observers u^α (i.e., the relative acceleration of two neighboring test particles of 4-velocity u^α , connected by X^α) vanishes. If X^α is orthogonal to the gravitomagnetic field, $H^\alpha X_\alpha = 0$, then it is an eigenvector of $\mathbb{E}_{\alpha\beta}$, of eigenvalue \vec{H}^2 . The eigenvectors X^α thus span a two dimensional subspace on the rest space $u^i = 0$, meaning that in these directions the tidal force is *proportional* to the separation vector X^α . Next we will physically interpret this for the special case of the Gödel universe.

4.1 The Gödel Universe

The Gödel universe is a solution corresponding to an *homogeneous* rotating dust with negative cosmological constant. The homogeneity requires that the dust rotates around every point. The line element is given by:

$$\mathcal{A}_i dx^i = e^{\sqrt{2}\omega x} dy , \quad \gamma_{ij} dx^i dx^j = dx^2 + \frac{1}{2} e^{2\sqrt{2}\omega x} dy^2 + dz^2 , \quad (105)$$

where ω is a constant. It is straightforward to show that the gravitomagnetic field \vec{H} is *uniform*, and equal to $\vec{H} = 2\omega \vec{e}_z$; hence, by virtue of (103), the gravitomagnetic tidal tensor vanishes: $\mathbb{H}_{\alpha\beta} = 0$. For this reason, this universe has been interpreted in [1, 2] as being analogous to an uniform magnetic field in the curved 3-manifold with metric γ , and the homogeneous rotation physically interpreted in analogy with a gas of charged particles subject to an uniform magnetic field — in that case one equally has Larmor orbits around any point.

Now we will interpret its gravitoelectric tidal tensor. In the coordinate system (105) it reads, for the $u^i = 0$ observers:

$$\mathbb{E}_{ij} = \omega^2 (\gamma_{ij} - \delta_i^z \delta_j^z) .$$

It vanishes along z , and is isotropic in the spatial directions x, y orthogonal to \vec{H} . It is similar to the Newtonian tidal tensor $\nabla_i \nabla_j V$ of a potential: $V = \omega^2 (x^2 + y^2)/2$, corresponding to a 2-D harmonic oscillator, which is the potential of the Newtonian analogue of the Gödel Universe [91]: an infinite cylinder of dust rotating rigidly with angular velocity ω . The potential V is such that the centrifugal force on each fluid element of the rigidly rotating cylinder exactly balances the gravitational attraction. This causes a curious effect in the Newtonian system: the fluid is seen to be rotating about any point at rest in the frame comoving with the “original” cylinder; indeed,

through an arbitrary point passes an axis of rotation relative to which the system is indistinguishable from the “original one”.

Therefore, whilst the gravitomagnetic field and tidal tensor, as well as the mapping via Klein-Gordon equation below, link to the magnetic analogue of the Gödel universe, the gravitoelectric tidal tensor links to the Newtonian analogue, both yielding consistent models to picture the homogeneous rotation of this universe.

5 Linear gravitoelectromagnetism

The oldest and best known gravito-electromagnetic analogies are the ones based on linearized gravity, which have been worked out by many authors throughout the years, see e.g. [23, 98, 99, 97, 64, 24, 70, 106, 35, 7]. In the way it is more usually presented, one considers a metric given by small perturbations $|h_{\alpha\beta}| \ll 1$ around Minkowski spacetime: $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, and from the components $h_{\alpha\beta}$ one defines the 3-vectors \vec{G} and \vec{H} , in terms of which the equations dubbed the metric is written in. The metric

$$ds^2 = -(1 + 2\Phi) dt^2 + 2\mathcal{A}_j dt dx^j + [\delta_{ij} + 2\xi_{ij}] dx^i dx^j, \quad (106)$$

If one considers stationary perturbations, as is more usual (e.g. [23, 70, 106, 35, 7]), the GEM fields are (up to numerical factors in the different definitions): $\vec{G} = -\nabla\Phi$, $\vec{H} = \nabla \times \vec{\mathcal{A}}$, where $\nabla_i \equiv \partial/\partial x_i$ denotes herein (and only herein!) the spatial derivative operator associated to the background euclidean metric δ_{ij} . These fields are straightforwardly related to the ones in Sec. 3: they are just, *to linear order*, minus the acceleration and twice the vorticity of the zero 3-velocity observers ($u^i = 0$) with respect to the coordinate system associated to the metric (106) (they can be called “static observers”). Thus they are simply a linear approximation to the Quasi-Maxwell fields in Eqs. (50). Note that, to linear order, $\tilde{\nabla}\Phi \approx \nabla\Phi$, $\tilde{\nabla} \times \vec{\mathcal{A}} \approx \nabla \times \vec{\mathcal{A}}$.

If the fields depend on time, different definitions of the fields exist in the literature, as a complete, one to one GEM analogy based on inertial fields, holding simultaneously for geodesics and for the field equations, is not possible, as we shall see below (cf. also [98, 97, 64, 2, 25]). So if one chooses to write one of them in an electromagnetic like form, the other will contain extra terms. We stick to defining \vec{G} and \vec{H} by minus the acceleration and twice the vorticity of the congruence (i.e. the same definitions given in Sec. 3.2 for congruence adapted frames, only this time linearized), which seems to make more sense physically, as with this definition the fields appear in the equation of geodesics playing formally analogous roles to the electric and magnetic fields in the Lorentz force. That amounts to define:

$$\vec{G} = -\nabla\Phi - \frac{\partial\vec{\mathcal{A}}}{\partial t}; \quad \vec{H} = \nabla \times \vec{\mathcal{A}}$$

The space part of the linearized equation for the geodesics, in the coordinate basis \mathbf{e}_i , is obtained from the corresponding exact equation in the tetrad field (45) of Sec. 3 as follows. Let $e_{\hat{\alpha}}^\alpha$ denote the transformation matrix between the coordinate basis $\mathbf{e}_\alpha \equiv \partial/\partial x_\alpha$ associated to the metric system in (106) and the orthonormal tetrad adapted to the $u^i = 0$ observers; that is, $\mathbf{e}_{\hat{\alpha}} = e_{\hat{\alpha}}^\alpha \mathbf{e}_\alpha$. To linear

order, $e^\alpha_{\hat{\alpha}}$, and its inverse $e^{\hat{\alpha}}_\alpha$, are given by:

$$\begin{aligned} \mathbf{e}_0 &= \frac{\mathbf{e}_0}{\sqrt{1+2\phi}}; & \mathbf{e}_i &= \mathbf{e}_i - \xi_i^j \mathbf{e}_j - A_i \mathbf{e}_0 \\ \mathbf{e}_0 &= \sqrt{1+2\phi} \mathbf{e}_0; & \mathbf{e}_i &= \mathbf{e}_i + \xi_i^j \mathbf{e}_j + A_i \mathbf{e}_0 \end{aligned}$$

Thus, $U^{\hat{i}} = e^{\hat{i}}_\alpha U^\alpha = U^i + \xi_i^j U_j$; putting also $U^i = dx^i/d\tau$, substituting in (45), linearizing in the perturbations and in v , and noting that, to linear order,

$$K_{(ij)} \equiv u_{(i;j)} = \sigma_{ij} + \frac{1}{3}\theta\delta_{ij} \approx \frac{\partial\xi_{ij}}{\partial t}; \quad \theta = K^i_i = \frac{\partial\xi^i_i}{\partial t} \quad (107)$$

the equation for the geodesics reads :

$$\frac{dx^2}{d\tau^2} = \vec{G} + \vec{v} \times \vec{H} - 2\frac{\partial\xi^i_j}{\partial t} v^j \vec{e}_i \quad (108)$$

That is, the extra term compared to the Lorentz force in electromagnetism comes *from the time derivative of the spatial metric* (which is true also in the exact case, as we have seen in Sec. 3.2). Noting that $d^2x/d\tau^2 \approx d^2\vec{x}/dt^2 - \vec{v}\partial\Phi/\partial t$, with $\vec{v} = d\vec{x}/dt$; we can also write this result as:

$$\frac{d^2\vec{x}}{dt^2} = \vec{G} + \vec{v} \times \vec{H} - 2\frac{\partial\xi^i_j}{\partial t} v^j \vec{e}_i + \frac{\partial\Phi}{\partial t} \vec{v} \quad (109)$$

The gravitational field equations in this regime are obtained by linearizing (73)-(78) and substituting relations (107):

$$\begin{aligned} \nabla \cdot \vec{G} &= -4\pi(2\rho + T^\alpha_\alpha) - \frac{\partial^2\xi^i_i}{\partial t^2}; & \text{(i)} \quad \nabla \times \vec{G} &= -\frac{\partial\vec{H}}{\partial t} & \text{(ii)} \\ \nabla \cdot \vec{H} &= 0; & \text{(iii)} \quad \nabla \times \vec{H} &= -16\pi\vec{J} + 4\frac{\partial}{\partial t}\xi_j^{[j,k]}\vec{e}_k & \text{(iv)} \\ G_{i,j} + \frac{1}{2}\epsilon_{ijk}\frac{\partial H^k}{\partial t} + \frac{\partial^2}{\partial t^2}\xi_{ij} + 2\xi^k_{(j,i)k} - \nabla^2\xi_{ij} - \xi^k_{k,ij} &= 8\pi\left(T_{ij} + \frac{1}{2}\delta_{ij}T^\alpha_\alpha\right) & \text{(v)} \end{aligned} \quad (110)$$

Eqs. (110i), (110iv), and (110v), are, respectively, the time-time, time-space, and space-space components of Einstein's equations with sources (14a); Eqs. (110iii) and (110) are, respectively the time-time and space-time components of the identities (14b). To obtain (110v) from the exact Eq. (72), we note that \tilde{R}_{ij} reads, to linear order

$$\tilde{R}_{ij} \simeq \Gamma_{ij,k}^k - \Gamma_{kj,i}^k \simeq 2\xi^k_{(j,i)k} - \nabla^2\xi_{ij} - \xi^k_{k,ij}$$

As for the time-space component of the identity (14b), i.e., Eq. (78), it yields, to linear order, the trivial equation $\star\tilde{R}^j_{ji} = 0$.

Eqs. (110) encompass two particularly important regimes: the ‘‘GEM limit’’, and gravitational radiation. Starting by the latter, in a source free region ($T^{\alpha\beta} = 0$) one can, as is well known, through gauge transformations (employing the harmonic gauge condition, and further specializing to the transverse traceless, or radiation, gauge, see e.g. [70]) make $\vec{\mathcal{A}} = \Phi = \xi^i_i = \xi^{ij}_{,j} = 0$; with this choice, the only non trivial equation left is (110v), yielding the 3-D wave equation $\partial^2\xi_{ij}/\partial t^2 = \nabla^2\xi_{ij}$. The GEM regime is obtained making $\xi_{ij} = -\Phi\delta_{ij}$ (which effectively neglects radiation); in this case

the traceless shear of the congruence of zero 3-velocity observers $u^i = 0$ (in the coordinates system of (106)) vanishes: $\sigma_{\alpha\beta} = 0$, and we have $u_{(i;j)} = \theta\delta_{ij}/3 = -\delta_{ij}\partial\Phi/\partial t$. This is the case also for the Post-Newtonian regime (e.g. [19, 47, 40, 116, 69]). The source is also assumed to be non-relativistic, so that the contribution of the pressure and stresses in Eq. (110) is negligible: $2\rho + T^\alpha_\alpha \approx \rho$. The equation for the geodesics (108) then reads

$$\frac{d\vec{U}}{dt} = \vec{G} + \vec{v} \times \vec{H} + 2\frac{\partial\Phi}{\partial t}\vec{v} \quad (111)$$

and Eqs. (110) above become

$$\begin{aligned} \nabla \cdot \vec{G} &= -4\pi\rho + 3\frac{\partial^2\Phi}{\partial t^2}; \quad (\text{i}) & \nabla \times \vec{G} &= -\frac{\partial\vec{H}}{\partial t} \quad (\text{ii}) \\ \nabla \cdot \vec{H} &= 0; \quad (\text{iii}) & \nabla \times \vec{H} &= -16\pi\vec{J} + 4\frac{\partial\vec{G}}{\partial t} - 4\frac{\partial^2\vec{A}}{\partial t^2} \quad (\text{iv}) \\ \frac{\partial}{\partial t}\mathcal{A}_{(i,j)} - \left(\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi\right)\delta_{ij} &= -4\pi\rho\delta_{ij} \quad (\text{v}) \end{aligned} \quad (112)$$

In some works, e.g. [24], the gravitoelectric field is given a different definition: $\vec{G}' = -\nabla\Phi - \frac{1}{4}\partial\vec{A}/\partial t$; with this definition, and choosing the harmonic gauge condition, which implies $\nabla \cdot \vec{A} = -4\partial\Phi/\partial t$, the non-Maxwellian term in Eq. (112i) disappears; but, on the other hand, a “non-Lorentzian” term appears in the equations for the geodesics, where in the place of \vec{G} in Eqs. (108)-(109), we would have instead $\vec{G}' - \frac{3}{4}\partial\vec{A}/\partial t$. The non-Maxwellian term in Eq. (112iv) is neglected in the Post Newtonian regime [47, 19].

The presence of the terms $\partial\vec{H}/\partial t$, “inducing” curls in \vec{G} and \vec{H} , respectively, analogous to the induction terms of electromagnetism, lead one to wonder if one can talk about gravitational induction effects in analogy with electrodynamics, and there is a debate concerning the applicability and physical content of this analogy for time-dependent fields, see e.g. [1] and references therein. Although a discussion of the approaches to this issue in the literature is not the scope of this work, still there are some points one can make based on the material herein. If one considers a time dependent gravitational field, such as the one generated by a moving point mass, e.g. Eq. (2.10) of [25], indeed the gravitoelectric field \vec{G} corresponding to this situation is different from the one for a point mass at rest, and has curl. That is, the acceleration $-\vec{G}$ of the congruence of observers at rest with respect to the background inertial frame (the “Post-Newtonian grid”, e.g. [40]), gains a curl when the source moves with respect to that frame. From Eq. (112ii), one can think about this curl as induced by the time-varying gravitomagnetic field \vec{H} , see e.g. [116]. These fields are well suited to describe the *apparent* Newtonian and Coriolis-like accelerations of particles in geodesic motion, as shown by Eq. (111) — above (one must only bear in mind that in the case of time-dependent fields, the motion is not determined solely by \vec{G} and \vec{H} ; there is an additional term with no analogue in the Lorentz force law, that leads to important significant differences). However the latter are mere artifacts of the reference frame; the *physical* (i.e., tidal) forces are a different story, one does not obtain the correct tidal forces by differentiation of these fields (as is the case with electrodynamics). Namely the curls of the GEM fields do not translate into these forces. The linearized gravito-electric tidal tensor, Eq. (95a), reads in the GEM regime ($K_{(ij)} = -\delta_{ij}\partial\Phi/\partial t$),

$$\mathbb{E}_{ij} \approx -G_{i,j} + \frac{1}{2}\epsilon_{ijk}\frac{\partial H^k}{\partial t} - \frac{\partial\Phi}{\partial t}\delta_{ij} = -G_{(i,j)} - \frac{\partial\Phi}{\partial t}\delta_{ij} \quad (113)$$

where we see that the curl (112ii) is subtracted from the derivative of \vec{G} . That is, only the symmetrized derivative $G_{(i,j)}$ describes physical, covariant forces. This is manifest in that the curl of \vec{G} does not induce a rotation on a set of neighboring particles (the gravitational field only shears them, see Sec. 2.2 and Eq. (26) therein), nor does it torque a rigid test body, see [4]. Note that in electromagnetism, this rotation and torque are tidal manifestations of Faraday's Law of induction. Likewise, the curl of \vec{H} is not manifest in the gravitomagnetic tidal effects (e.g., the force on a gyroscope); the linearized gravitomagnetic tidal tensor (95b) reads, in this regime:

$$\mathbb{H}_{ij} \approx -\frac{1}{2} \left[H_{i,j} - 2\epsilon_{ijl} \left(\frac{\partial G^l}{\partial t} - \frac{\partial^2 \mathcal{A}^l}{\partial t^2} \right) \right] \quad (114)$$

where again we can see that the induction contribution $4\partial\vec{G}/\partial t$ (and also the one of the term $\partial^2\vec{A}/\partial t^2$) to the curl of \vec{H} is subtracted from the derivative of \vec{H} . The physical consequences are explored in [4]: in electromagnetism, due to vacuum equation $\nabla \times \vec{B} = -\partial\vec{E}/\partial t$, there is a non-vanishing force on a magnetic dipole, $F_{EM}^i = B^{ji}\mu_j = \nabla^i(\vec{\mu} \cdot \vec{B})$, whenever it moves in a non-homogeneous field (since the electric field it measures is time-varying, thus $\nabla \times \vec{B} \neq 0 \Rightarrow \vec{F}_{EM} \neq 0$). This expression for F_{EM}^i holds *in the rest frame of the particle*, for arbitrary fields. That is not necessarily the case in gravity. In vacuum, from Eqs. (112iv) and (114), we have $\mathbb{H}_{ij} = -H_{(i,j)}/2$, and the gravitational force on a gyroscope, cf. Eq. (1.2b) of Table 1, is $F_G^i = \frac{1}{2}H^{(i,j)}S_j$. Thus no analogous induction effect is manifest in the force, and in fact spinning particles in non-homogeneous gravitational fields can move along geodesics, which is exemplified in [4].

In the case that the field is stationary, we have a one to one correspondence with electromagnetism *in inertial frames*. Eq. (v) above becomes identical to (i), and then we are left with a set of four equations — Eqs. (110i)-(110iv) with the time dependent terms dropped — similar, up to some factors, to the time-independent Maxwell equations in an inertial frame. These equations can also be obtained by linearization of Eqs. (34b)-(34b) of Table 3. The space part of the equation of the geodesics: $dU/dt = \vec{G} + \vec{v} \times \vec{H}$, cf. Eq. (108) above, is also similar to the Lorentz force in a Lorentz frame. The equation for the evolution of the spin vector of a gyroscope becomes simply $d\vec{S}/d\tau = \vec{S} \times \vec{H}/2$, which gives the precession relative to the background Minkowski frame, and is similar to the precession of a magnetic dipole in a magnetic field. The force on a gyroscope whose center of mass it *at rest* is $\vec{F}_G = \nabla(\vec{S} \cdot \vec{H})/2$, similar to the force $\vec{F}_{EM} = \nabla(\vec{\mu} \cdot \vec{B})$ on a magnetic dipole at rest in a Lorentz frame; the same for the differential precession of gyroscopes/dipoles at rest, they read respectively, for a spatial separation vector δx^α , $\delta\Omega_G = -\nabla(\delta\vec{x} \cdot \vec{H})/2$, $\delta\Omega_{EM} = -\nabla(\delta\vec{x} \cdot \vec{B})$.

6 The *formal* analogy between gravitational *tidal tensors* and electromagnetic *fields*

There is a set of analogies, based on exact expressions, relating the Maxwell tensor $F^{\alpha\beta}$ and the Weyl tensor $C_{\alpha\beta\gamma\delta}$, namely: 1) they both irreducibly decompose in two electric and magnetic type spatial tensors; these tensors obey differential equations — Maxwell's equations and the so called “higher order” gravitational field equations — which are formally analogous to a certain extent [30, 31, 85, 32, 51], and they form, moreover, invariants in a similar fashion to the relativistic invariants formed by the electric and magnetic fields [30, 31, 52, 53]. In this section we will briefly review this analogy and clarify its physical content in the light of the previous approaches.

The Maxwell tensor splits, with respect to a unit time-like vector U^α , in its electric and magnetic parts:

$$E^\alpha = F^\alpha{}_\beta U^\beta, \quad B^\alpha = \star F^\alpha{}_\beta U^\beta, \quad (115)$$

i.e., the electric and magnetic fields as measured by the observers of 4-velocity U^α . These are spatial vectors: $E^\alpha U_\alpha = B^\alpha U_\alpha = 0$, thus possessing 3+3 independent components, which completely encode the characterize the 6 independent components of $F_{\mu\nu}$, as can be seen explicitly in decompositions (1). In spite of their dependence on U^α , one can use E_α and B_α to define two tensorial quantities which are U^α independent, namely

$$E^\alpha E_\alpha - B^\alpha B_\alpha = -\frac{F_{\alpha\beta} F^{\alpha\beta}}{2}, \quad E^\alpha B_\alpha = -\frac{F_{\alpha\beta} \star F^{\alpha\beta}}{4}; \quad (116)$$

these are the only algebraically independent invariants one can define from the Maxwell tensor.

The Weyl tensor has a formally similar decomposition: with respect to a unit time-like vector U^α , it splits irreducibly in its electric $\mathcal{E}_{\alpha\beta}$ and magnetic $\mathcal{H}_{\alpha\beta}$ parts:

$$\mathcal{E}_{\alpha\beta} \equiv C_{\alpha\gamma\beta\sigma} U^\gamma U^\sigma, \quad \mathcal{H}_{\alpha\beta} \equiv \star C_{\alpha\gamma\beta\sigma} U^\gamma U^\sigma. \quad (117)$$

These two spatial tensors, both of which are symmetric and traceless (hence have 5 independent components each), completely encode the 10 independent components of the Weyl tensor, as can be seen by writing [32]

$$C_{\alpha\beta}{}^{\gamma\delta} = 4 \left\{ 2U_{[\alpha} U^{[\gamma} + g_{[\alpha}^{[\gamma} \right\} \mathcal{E}_{\beta]}^{\delta]} + 2 \left\{ \epsilon_{\alpha\beta\mu\nu} U^{[\gamma} \mathcal{H}^{\delta]\mu} U^\nu + \epsilon^{\gamma\delta\mu\nu} U_{[\alpha} \mathcal{H}_{\beta]\mu} U_\nu \right\}. \quad (118)$$

Again, in spite of their dependence on U^α , one can use $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$ to define the two tensorial quantities which are U^α independent

$$\mathcal{E}^{\alpha\beta} \mathcal{E}_{\alpha\beta} - \mathcal{H}^{\alpha\beta} \mathcal{H}_{\alpha\beta} = \frac{C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu}}{8}, \quad \mathcal{E}^{\alpha\beta} \mathcal{H}_{\alpha\beta} = \frac{C_{\alpha\beta\mu\nu} (\star C)^{\alpha\beta\mu\nu}}{16}. \quad (119)$$

which are *formally* analogous to the electromagnetic scalar invariants (116). However, by contrast with the latter, these are not the only independent scalar invariants one can construct from $C_{\alpha\beta\mu\nu}$; there are also two cubic invariants, see [63, 31, 54, 55].

As stated above, these tensors obey also differential equations which have some formal similarities with Maxwell's; such equations, dubbed the “higher order field equations” are obtained from Bianchi identities $R_{\sigma\tau[\mu\nu;\alpha]} = 0$. These, together with field equations (14a), lead to:

$$C^\mu{}_{\nu\sigma\tau;\mu} = 8\pi \left(T_{\nu[\tau;\sigma]} - \frac{1}{3} g_{\nu[\tau} T_{;\sigma]} \right), \quad (120)$$

Expressing $C_{\alpha\beta\delta\gamma}$ in terms of $\mathcal{E}_{\alpha\beta}$, $\mathcal{H}_{\alpha\beta}$ using (118), and taking time and space projections of (120) using the projectors (2), we obtain, assuming a *perfect fluid*, the set of equations

$$\begin{aligned} \tilde{\nabla}^\mu \mathcal{E}_{\nu\mu} &= \frac{8\pi}{3} \tilde{\nabla}_\nu \rho + 3\omega^\mu \mathcal{H}_{\nu\mu} + \epsilon_{\nu\alpha\beta} \sigma^\alpha{}_\gamma \mathcal{H}^{\beta\gamma}, \\ \text{curl} \mathcal{H}_{\mu\nu} &= \frac{D_F}{d\tau} \mathcal{E}_{\mu\nu} + \mathcal{E}_{\mu\nu} \theta - 3\sigma_{\tau\langle\mu} \mathcal{E}_{\nu\rangle}^\tau - \omega^\tau \epsilon_{\tau\rho(\mu} \mathcal{E}_{\nu)}^\rho - 2a^\rho \epsilon_{\rho\tau(\mu} \mathcal{H}_{\nu)}^\tau + 4\pi(\rho + p) \sigma_{\mu\nu}, \end{aligned} \quad (121)$$

$$\tilde{\nabla}^\mu \mathcal{H}_{\nu\mu} = 8\pi(\rho + p)\omega_\nu - 3\omega^\mu \mathcal{E}_{\nu\mu} - \epsilon_{\nu\alpha\beta} \sigma^\alpha_\gamma \mathcal{E}^{\beta\gamma}, \quad (122)$$

$$\text{curl} \mathcal{E}_{\mu\nu} = \frac{D_F}{d\tau} \mathcal{H}_{\mu\nu} - \mathcal{H}_{\mu\nu} \theta + 3\sigma_{\tau\langle\mu} \mathcal{H}_{\nu\rangle}^\tau + \omega^\tau \epsilon_{\tau\rho(\mu} \mathcal{H}_{\nu)}^\rho - 2a^\rho \epsilon_{\rho\tau(\mu} \mathcal{E}_{\nu)}^\tau.$$

where $\epsilon_{\mu\nu\rho} \equiv \epsilon_{\mu\nu\rho\tau} U^\tau$, $\tilde{\nabla}$ denotes the spatially projected covariant derivative, cf. Eq. (56), $\text{curl} A_{\alpha\beta} \equiv \epsilon^{\mu\nu}_{(\alpha} A_{\beta)\nu;\mu}$, and the index notation $\langle\mu\nu\rangle$ stands for the spatially projected, symmetric and trace free part of a rank two tensor (cf. definitions in [32]):

$$A_{\langle\mu\nu\rangle} \equiv h_{(\mu}^\alpha h_{\nu)} A_{\alpha\beta} - \frac{1}{3} h_{\mu\nu} h_{\alpha\beta} A^{\alpha\beta};$$

θ , $\sigma_{\mu\nu} \equiv D_{\langle\mu} U_{\nu\rangle}$, $\omega^\alpha \equiv \epsilon^\alpha_{\beta\gamma} U_{\gamma;\beta}/2$, a^α are, respectively, the expansion, shear, vorticity, and acceleration of the congruence of observers with 4-velocity U^α .

The analogous electromagnetic equations are the ones in Sec. 3.4.1,

$$\tilde{\nabla}_\mu E^\mu = 4\pi\rho_c + 2\omega_\mu B^\mu \quad (123)$$

$$\epsilon^{\alpha\gamma\beta} B_{\beta;\gamma} = \frac{D_F E^\alpha}{d\tau} - \sigma^\alpha_\beta E^\beta + \frac{2}{3} \theta E^\alpha - \epsilon^\alpha_{\beta\gamma} \omega^\beta E^\gamma + \epsilon^\alpha_{\beta\gamma} B^\beta a^\gamma + 4\pi j^{\langle\alpha} \quad (124)$$

$$\tilde{\nabla}_\mu B^\mu = -2\omega_\mu E^\mu \quad (125)$$

$$\epsilon^{\alpha\gamma\beta} E_{\beta;\gamma} = -\frac{D_F B^\alpha}{d\tau} + \sigma^\alpha_\beta B^\beta - \frac{2}{3} \theta B^\alpha + \epsilon^\alpha_{\beta\gamma} \omega^\beta B^\gamma + \epsilon^{\alpha\mu\sigma} E_\mu a_\sigma \quad (126)$$

Eqs. (123) and (125) follow from Eqs. (54) and (58), respectively, by noting that, for an arbitrary vector A^α ,

$$A^\beta_{;\beta} = \left(\mathbb{T}^\gamma_\beta + h^\gamma_\beta \right) \left(\mathbb{T}^\beta_\lambda + h^\beta_\lambda \right) A^\lambda_{;\gamma} = \left(\mathbb{T}^\gamma_\lambda + h^\gamma_\lambda \right) A^\lambda_{;\gamma} = A^\beta a_\beta + \tilde{\nabla}_\alpha A^\alpha.$$

Eqs. (124) and (126) follow from Eqs. (60) and (64) by decomposing $K_{(\alpha\beta)} = \sigma_{\alpha\beta} + \theta h_{\alpha\beta}/3$.

6.1 Matte's equations vs Maxwell equations. Tidal tensor interpretation of gravitational radiation.

Table 4: Formal analogy between Maxwell equations (differential equations for electromagnetic *fields*) and Matte's equations (differential equations for gravitational *tidal* tensors)

Electromagnetism		Linearized Gravity	
Maxwell's Equations		Matte's Equations	
$E^i_{,i} = 0$	(4.1a)	$\mathbb{E}^{ij}_{,i} = 0$	(4.1b)
$B^i_{,i} = 0$	(4.2a)	$\mathbb{H}^{ij}_{,i} = 0$	(4.2b)
$\epsilon^{ikl} E_{l,k} = -\frac{\partial B^i}{\partial t}$	(4.3a)	$\epsilon^{ikl} \mathbb{E}^j_{l,k} = -\frac{\partial \mathbb{H}^{ij}}{\partial t}$	(4.3b)
$\epsilon^{ikl} B_{l,k} = \frac{\partial E^i}{\partial t}$	(4.4a)	$\epsilon^{ilk} \mathbb{H}^j_{l,k} = \frac{\partial \mathbb{E}^{ij}}{\partial t}$	(4.4b)
Wave equations		Wave equations	
$\left(\frac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) E^i = 0$	(4.5a)	$\left(\frac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) \mathbb{E}_{ij} = 0$	(4.5b)
$\left(\frac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) B^i = 0$	(4.6a)	$\left(\frac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) \mathbb{H}_{ij} = 0$	(4.6b)

In vacuum, the Bianchi identities become:

$$R_{\sigma\tau[\mu\nu;\alpha]} = 0; \quad (a) \quad R^\mu_{\alpha\beta\gamma;\mu} = 0 \quad (b) \quad (127)$$

(the second equation following from the first and from vacuum equation $R_{\mu\nu} = 0$). The formal analogy with Eqs. (6), for $j^\alpha = 0$, is now more clear [31]. In a nearly Lorentz frame, and to linear order in the metric potentials, Eqs. (121)-(122), for vacuum, become Eqs. (4.1b)-(4.4b) of Table 4, which are *formally* similar to Maxwell's equations in a Lorentz frame (4.1a)-(4.4a). The analogy in Eqs. (4.1)-(4.4) was first found by Matte [30], and further studied by some other authors⁸ [31, 61, 62]. Taking curls of Eqs. (4.3a)-(4.4a) we obtain the wave equations for the electromagnetic fields; and taking curls of (4.3a)-(4.4a), we obtain gravitational waves, as wave equations for gravitational tidal tensors.

Hence, to this degree of accuracy, vacuum gravitational waves can be cast as a pair of oscillatory tidal tensors $\mathbb{E}_{\alpha\beta}$, $\mathbb{H}_{\alpha\beta}$ propagating in space by mutually inducing each other, just like the pair of fields E^α , B^α in the case of the electromagnetic waves. Also, just like E^α and B^α are equal in magnitude and mutually orthogonal for a purely radiative field, the same applies to the waves in (4.5b)-(4.6b) of Table 4. In the electromagnetic case this implies that the two invariants (116) vanish; likewise, the gravitational invariants (119) also vanish for a solution corresponding to pure

⁸The exact equations (121)-(122), in vacuum, can take the form (4.1b)-(4.4b) of Table 4, in a *local* Lorentz frame, where $a^\alpha = \omega^\alpha = \sigma_{\alpha\beta} = \theta = 0$, as done in e.g. [102]. That however holds only in a small neighborhood around a particular observer worldline, and is therefore not suited to study the propagation of radiation.

gravitational radiation according to Bel’s second criterion (cf. e.g. [60] p. 53) — a definition based on “super-energy”, discussed below.

An interesting aspect of this formulation of gravitational radiation, contrasting with the more usual approaches in the literature, e.g. [70, 28, 23, 3] — which consist of equations for the propagation of gauge fields (the components of the metric tensor) having no local physical significance (only their second derivatives may be related to physically measurable quantities, see in this respect [75]) — is that Eqs. (4.5b)-(4.6b) are equations for the propagation of *tensors of physical forces*, with direct translation in physical effects – the relative acceleration of two neighboring test particles via geodesic deviation equation (1.1b), and the force on a spinning test particle, via Mathisson-Papapetrou-Pirani Eq. (1.2b), or the relative precession of two nearby gyroscopes, via Eq. (27). The latter effects come from the “magnetic part” (i.e., involving $\mathbb{H}_{\alpha\beta}$) of the radiation, little-studied in comparison to its electric counterpart; its role, in equal footing with the electric part, is made explicit in this formulation — they are both essential for the existence of gravitational waves, as pair of mutually inducing fields is needed. It yields a simple and intuitive description of the interaction of a gravitational wave with a pole-dipole spinning particle, as a coupling of the wave equation (4.6b) to the spin-vector of the particle, via Eq. (1.2b). In other words: putting together Matte’s equations with the physical interpretation of the electric and magnetic parts of the Riemann tensor given in Sec. 2, we obtain this suggestive interpretation of gravitational radiation, which has not (to our knowledge) been put forth in the existing literature. That might be down to two reasons: on the one hand the works treating the higher order/Matte equations [30, 31, 75, 85, 61, 62, 51, 87], as well the tensors $\{\mathcal{E}_{\alpha\beta}, \mathcal{H}_{\alpha\beta}\}$ in other contexts e.g. [84, 87, 56], lack a physical interpretation of the tensors $\mathcal{H}_{\alpha\beta}/\mathbb{H}_{\alpha\beta}$ (the tensors $\mathcal{E}_{\alpha\beta}/\mathbb{E}_{\alpha\beta}$ in their turn are reasonably well understood due to their role in the geodesic deviation equation), which are portrayed as not well understood [84, 87, 56], or given inconsistent interpretations that immediately lead to contradictions [82, 83, 32] (see [2] for more details). On the other hand, in the literature concerning the interaction of gravitational waves with spinning particles, e.g. [72, 71, 73, 74], neither gravitational waves are written in the form (4.5b)-(4.6b), nor, as a matter of fact, is the force on a spinning particle explicitly related to the magnetic part of the Riemann tensor $\mathbb{H}_{\alpha\beta}$ (or to $\mathcal{H}_{\alpha\beta}$), which is likely down to the fact that in these treatments the Tulczyjew-Dixon spin condition $S^{\alpha\beta}P_\beta = 0$ is employed in the Mathisson-Papapetrou equation (first term of Eq. (130)), instead of the Mathisson-Pirani condition $S^{\alpha\beta}U_\beta = 0$. The two conditions are equally valid (see in this respect [5]); however it is only when one uses the latter, that one obtains the expression (1.2b) for the force (see [4] for more details).

It is also interesting to note that in the traditional treatments the wave equations are obtained from a linearized form of Einstein equations (14a); whereas equations (4.5b)-(4.6b) come from a linearization of the higher order field equations (127b). (Even though the former still play a role, as in order to obtain Eqs. (4.5b)-(4.6b) from the differential Bianchi identity (127a), we have substitute $R_{\alpha\beta}$ by the source terms using Einstein’s equations (14a)).

It is important to realize that whereas in the electromagnetic radiation are the vector fields that propagate, gravitational radiation is a purely tidal effect, i.e., traveling tidal tensors not subsidiary to any associated (electromagnetic-like, or Newtonian-like) vector “field”; it is well know that there are no vector waves in gravity (see e.g. [65, 28, 64]; such waves would carry negative energy if they were to exist, cf. [28] p. 179). We have seen in Sec. 3.5, except for the very special cases of

the linear regime in weak, stationary fields (and static observers therein), the gravitational tidal tensors cannot be cast as derivatives of some vector field. In the electromagnetic case of course there are also tidal effects associated to the wave; but their dynamics follows trivially⁹ from Eqs.(4.3a)-(4.4a); to this accuracy, the tidal tensors as measured by the background static observers are just $E_{ij} = E_{i,j}$, $B_{ij} = B_{i,j}$; hence the equations of their evolution (i.e., the “electromagnetic higher order equations”) are:

$$\epsilon_i^{kl} E_{jl,k} = 0 \quad \epsilon_i^{kl} B_{jl,k} = 0 \quad (128)$$

$$\epsilon_i^{kl} E_{lj,k} = \epsilon_i^{kl} E_{lk,j} = -\frac{\partial B_{ij}}{\partial t} \quad \epsilon_i^{kl} B_{lj,k} = \epsilon_i^{kl} B_{lk,j} = \frac{\partial E_{ij}}{\partial t} \quad (129)$$

These four equations are the *physical* analogues of the pair of gravitational Eqs. (4.3b)-(4b); we have two more equations in electromagnetism since E_{ij} and B_{ij} are not symmetric. Eqs. (128), and the first equality in Eqs. (129), come from the fact that derivatives in flat spacetime commute; therefore $\epsilon_i^{kl} E_{jl,k} = \epsilon_i^{kl} E_{j,[lk]} = 0$ and $E_{lj,k} = E_{l,jk} = E_{lk,j}$. Thus, Eqs. (129), which are the only ones that contain dynamics, are obtained by simply differentiating ∂/∂_j Eqs. (4.3a)-(4.4a). And the wave equations for the electromagnetic tidal tensors follow likewise from differentiating ∂/∂_j Eqs. (45a)-(46a). Note also that the fact that in gravity, $\mathbb{H}_{j[l,k]} \neq 0$ is again related to the fact that, even in the linear regime, the gravitational tidal tensors are not derivatives of some vector fields.

The tidal tensor interpretation of gravitational waves gives insight into some of their fundamental aspects; we will mention three. Firstly, being gravitational waves (in vacuum) traveling tidal tensors, then according to discussion in 2, they couple to dipole particles (i.e., spinning particles, or “gravitomagnetic dipoles”) causing a force, and they can also cause a relative acceleration between two neighboring monopole particles; but they cannot exert any force on a monopole test particle (by contrast with their electromagnetic counterparts); that is the reason why a gravitational wave distorts a (approximately monopole) test body, but does not displace its mass center, as is well known, e.g. [23]. Let us state this important point in terms of rigorous equations. As explained in Sec. 2, an extended test body may be represented by its electromagnetic and “gravitational” multipole moments (i.e., the moments of j_p^α and $T_p^{\alpha\beta}$, respectively). The force exerted on a charged body in a electromagnetic field is, up to quadrupole order [9, 35, 4]:

$$\frac{D(P_{can})^\alpha}{d\tau} = qF^{\alpha\beta}U_\beta + \frac{1}{2}F^{\mu\nu;\alpha}Q_{\mu\nu} + \frac{1}{3}Q_{\beta\gamma\delta}F^{\gamma\delta;\beta\alpha} + \dots$$

where U^α is the particle’s 4-velocity, q is the charge, $Q_{\alpha\beta}$ is the dipole moment 2-form, which we may write as $Q_{\alpha\beta} = 2d_{[\alpha}U_{\beta]} + \epsilon_{\alpha\beta\gamma\delta}\mu^{\gamma}U^{\delta}$, and $Q_{\alpha\beta\gamma}$ is a quadrupole moment of the charge 4-current density j^α . Using decomposition (7), we can write the second term in terms of tidal tensors ($F^{\mu\nu;\alpha}Q_{\mu\nu}/2 = E^{\beta\alpha}d_\beta + B^{\beta\alpha}\mu_\beta$), the third term in terms of derivatives of the tidal tensors, and so on. Thus we see that in the electromagnetic case, there are force terms coming from the electromagnetic monopole, dipole, quadrupole moments, etc, corresponding to the coupling to, respectively, the field, the tidal field, derivatives of tidal field, etc.

The *exact* equations of motion for a test body in a gravitational field are [8, 35], up to quadrupole order:

$$\frac{DP^\alpha}{d\tau} = -\frac{1}{2}R^\alpha_{\beta\mu\nu}S^{\mu\nu}U^\beta - \frac{1}{6}J_{\beta\gamma\delta\sigma}R^{\beta\gamma\delta\sigma;\alpha} + \dots \quad (130)$$

⁹We thank J. Penedones for discussions on this point.

where $S_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} S^\gamma U^\delta$ is the spin 2-form, which is the dipole moment of the mass current 4-vector $J_p^\alpha = -T_p^{\alpha\beta} U_\beta$, see e.g. [9, 4], and $J_{\beta\gamma\delta\sigma}$ is a quadrupole moment of the energy-momentum tensor of the test particle, $T_p^{\alpha\beta}$. The first term yields of course $-\mathbb{H}^{\beta\alpha} S_\beta$, in agreement with Eq. (1.2b), corresponding to the coupling of the gravitomagnetic tidal tensor with the spin vector S^α (the gravitomagnetic dipole moment), the second term the coupling of the derivatives of the tidal field to the quadrupole moment, and so on. Hence we see that, by contrast with electromagnetism, in the gravitational case the interaction starts only at dipole order; as the lowest order (in differentiation) *physical* fields present in gravity are the tidal tensors (the real, physical content of the gravitational waves), which do not couple to monopoles.

Secondly, it sheds also some light on the issues of gravitational induction. As seen in Secs. 2, 3.5, and 5, electromagnetic-like induction effects are absent in the gravitational physical forces. However, instead of some vector fields, in gravity one can talk of a different type of induction phenomena, *for tidal tensors*: a time-varying magnetic tidal tensor induces an electric tidal tensor, and vice-versa, as implied by Eqs. (4.3b)-(4.4b). These tidal tensors propagate by mutually inducing each other, giving rise to the gravitational radiation, just like the laws of electromagnetic induction lead to electromagnetic radiation.

Finally, one can also relate the fact that in the gravitational waves what propagates are the tidal tensors, without any associated electromagnetic like vector field, with the fact that, to the gravitational waves (and the gravitational field itself), one cannot associate a *local* energy or momentum density, in the traditional sense of being manifest in the energy momentum tensor $T^{\alpha\beta}$. This becomes more clear contrasting with the electromagnetic case; the electromagnetic field gives a contribution to the total energy momentum tensor $T^{\alpha\beta}$ equal to:

$$T_{EM}^{\alpha\beta} = \frac{1}{8\pi} \left[F^{\alpha\gamma} F^\beta_\gamma + \star F^{\alpha\gamma} \star F^\beta_\gamma \right] \quad (131)$$

leading to an energy density ρ_{EM} and spatial momentum density $p_{EM}^{(\alpha)}$ (Poynting vector), with respect to an observer of 4-velocity u^α :

$$\rho_{EM} \equiv T_{EM}^{\alpha\beta} u_\alpha u_\beta = \frac{1}{8\pi} [(E^u)^\alpha (E^u)_\alpha + (B^u)^\alpha (B^u)_\alpha] \quad (132)$$

$$p_{EM}^{(\alpha)} \equiv -T_{EM}^{(\alpha)\beta} u_\beta = \frac{1}{4\pi} \epsilon^\alpha_{\mu\nu\sigma} u^\sigma (E^u)^\mu (B^u)^\nu \quad (133)$$

or, in vector notation, $8\pi\rho_{EM} = \vec{E}(u)^2 + \vec{B}(u)^2$; $\vec{p}_{EM} = \vec{E}(u) \times \vec{B}(u)/4\pi$. Now note that the significance of these expressions as energy and momentum densities can be traced back [76, 26], at the most fundamental level, to the Lorentz force (together with Maxwell equations), the work done by it, and its spatial momentum transfer. It is thus clear that it cannot have a direct *physical* gravitational analogue, as there is no physical, covariant, gravitational counterpart to the Lorentz force and the vector fields \vec{E} , \vec{B} ; the inertial “force” (48), and its Lorentz-like form for stationary spacetimes (48) (both being but the geodesic equation in a different language, no *real* force being involved), as well as the fields \vec{G} and \vec{H} , are mere artifacts of the reference frame, which can be made to vanish by switching to a locally inertial frame, as explained in Sec. 3. It is the same for the “energy” and “momentum densities” arising from the Landau-Lifshitz pseudo-tensor $t_{\mu\nu}$, e.g. [77, 28]; the “momentum density” arising from $t_{\mu\nu}$, to Post-Newtonian order, may actually be written as [45]: $\vec{p}_G \approx (-\vec{G} \times \vec{H} + 3\vec{G}\dot{\Phi})/4\pi$

6.1.1 Super-energy

On the other hand there is a quantity built on tidal tensors (not on vector fields), having thereby a local physical existence, that seems to fit well in the tidal tensor interpretation of gravitational radiation — the so-called Bel-Robinson super-energy tensor [31], which reads in vacuum:

$$T^{\alpha\beta\gamma\delta} = \frac{1}{2} \left[R^{\alpha\rho\gamma\sigma} R^{\beta}_{\rho}{}^{\mu}_{\sigma} + \star R^{\alpha\rho\gamma\sigma} \star R^{\beta}_{\rho}{}^{\mu}_{\sigma} \right] \quad (134)$$

(the more general expression in the presence of sources is given in e.g. [78]; a general superenergy tensor can also be defined, e.g. [32, 78] from the Weyl tensor; it is obtained by simply replacing $R_{\alpha\beta\gamma\delta} \rightarrow R_{\alpha\beta\gamma\delta}$). The *formal* analogy to the electromagnetic energy-momentum tensor, Eq. (131) above, is clear. The super-energy density and flux are defined as:

$$W \equiv T^{\alpha\beta\gamma\delta} u_{\alpha} u_{\beta} u_{\gamma} u_{\delta} = \frac{1}{2} \left[(\mathbb{E}^u)^{\alpha\beta} (\mathbb{E}^u)_{\alpha\beta} + (\mathbb{H}^u)^{\alpha\beta} (\mathbb{H}^u)_{\alpha\beta} \right] \quad (135)$$

$$\mathcal{P}^{(\alpha)} \equiv -T^{(\alpha)\beta\gamma\delta} u_{\beta} u_{\gamma} u_{\delta} = \epsilon^{\alpha}_{\mu\nu\sigma} u^{\sigma} (\mathbb{E}^u)^{\mu\lambda} (\mathbb{H}^u)^{\nu}_{\lambda} \quad (136)$$

where the gravitational tidal tensors play a role *formally* analogous to the electromagnetic fields in the quantities ρ_{EM} and $p_{EM}^{(\alpha)}$, Eqs. (132)-(133) above. Note that expressions (135)-(136) are *exact*. W is positive definite, and is zero only if the curvature vanishes [31, 60]; it thus qualitatively agrees with one would expect from field energy associated with the *physical* gravitational field (i.e., the curvature tensor). According to Bel [31, 60], gravitational radiation is characterized by a flux of super-energy \mathcal{P}^{α} (which, at least in the linear regime, is clear from the equations (4.1b)-(4.6b) above), parallel to the spatial direction of propagation of the wave. And that, at a given point, the non-vanishing of $\mathcal{P}^{(\alpha)}$ for *every* observer u^{α} (i.e., an *intrinsic* super-flux), is sufficient to ensure that gravitational radiation is present. Several criteria¹⁰ have been proposed (see [110] for a review) to characterize radiative states, based on the electromagnetic analogy. It is indeed tempting to think of super-energy and super-momentum (or super-flux) as the form in which gravitational waves carry what then, *in the interaction with matter* — e.g., through the test body multipole moments, as described by Eq. (130) — manifests itself as ordinary energy and momentum. This is an hypothesis to be studied in detail elsewhere; herein we would just like to point out that this is line with the point of view in [75] (where it is also shown that, via W , gravitational radiation has an active attractive effect), and that the fact that super-energy is not a conserved quantity in general in the presence of matter [79, 78], seems also to strenghten the hypothesis. There are however conceptual difficulties [78, 81, 109] in the physical interpretation of these quantities, such as the “strange” dimensions of W , which are L^{-4} , that (taking $L = M$) can be interpreted an energy density per unit area [49, 78, 79], or of an energy density times a frequency squared [75], or an energy density squared [78]. The very question if the superenergy has any physical reality is an open question, as is still the old problem of the definition of the energy of the gravitational field (which we will not try to further elaborate herein). In the spirit of the arguments given above — that the GEM fields are frame artifacts, the non-existence of local quantities *physically* analogous to the electromagnetic vector fields (consequence of the equivalence principle), and that gravitational waves are tidal tensors propagating by mutually inducing each other (without associated vector fields), which do not interact with the multipole structure of test bodies in the same way electromagnetic waves do

¹⁰We thank A. García-Parrado for discussions on this issue.

(in particular, do not couple to monopole particles) — it seems plausible that the dimensions of the fundamental “entity” carried by gravitational waves are not the same of their electromagnetic counterparts, hence not the ones of an energy. This is consistent with the point of view in [49, 109].

6.2 The relationship with the other GEM analogies

The analogy drawn in this section is between the electromagnetic fields and the electric and magnetic parts of the Weyl tensor: $\{E^\alpha, B^\alpha\} \leftrightarrow \{\mathcal{E}^{\mu\nu}, \mathcal{H}^{\mu\nu}\}$. It is clear, from the discussion of the physical meaning of $\{\mathbb{E}^{\mu\nu}, \mathbb{H}^{\mu\nu}\}$ in Sec. 2, and from the dynamical gravitational counterparts of $\{E^\alpha, B^\alpha\}$ discussed in Sec. 3, that this analogy is a purely formal one. It draws a parallelism between electromagnetic fields, whose dynamical gravitational analogues are the GEM inertial fields $\{\vec{G}, \vec{H}\}$ of Sec. 3, with gravitational *tidal* fields, which are the physical analogues *not* of $\{E^\alpha, B^\alpha\}$, but of the electromagnetic *tidal* tensors $\{E_{\alpha\beta}, B_{\alpha\beta}\}$ (which, for an observer of fixed 4-velocity U^α , are covariant derivatives of the E^α and B^α it measures). This sheds light on some conceptual difficulties in the literature regarding the physical content of the analogy and in particular the physical interpretation of the tensor $\mathcal{H}^{\mu\nu}$, see [2] for details. And is also of crucial importance for the correct understanding of physical meaning of the curvature invariants, and their implications on the motion of test particles, which is subject of detailed study in [63].

7 When can gravity be similar to electromagnetism?

A crucial point to realize is that the two exact physical gravito-electromagnetic analogies — the tidal tensor analogy of Sec. 2 and the inertial GEM fields analogy of Sec. 2 — do not rely on a close physical similarity between the interactions; as the gravitational objects $\{\vec{G}, \vec{H}, \mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta}\}$, despite playing *analogous dynamical roles* to the objects $\{\vec{E}, \vec{B}, E_{\alpha\beta}, B_{\alpha\beta}\}$ in electromagnetism, are themselves in general very different from the latter. Even for seemingly analogous setups (e.g. the EM field of spinning charge, and the gravitational field of a spinning mass). In this sense these analogies have a different status compared to the popular GEM analogy based on linearized theory, which, *in order to hold*, indeed requires a close similarity between the two interactions, to which the former two are not bound.

What the tidal tensor formalism of Sec. 2, together with the inertial fields formalism of Sec. 3, provide, is a “set of tools” to determine under which precise conditions a similarity between the gravitational and electromagnetic interactions may be expected.

The key differences between electromagnetic and gravitational tidal tensors are: a) they do not exhibit, generically, the same symmetries; b) gravitational tidal tensors are spatial whereas electromagnetic ones are not; c) electromagnetic tidal tensors are linear, whereas the gravitational ones are not.

The electromagnetic tidal tensors, for a given observer, only have the same symmetries and time-projections as the gravitational ones when the Maxwell tensor is covariantly constant along the observer’s worldline; that is implied by Eqs. (1.4) and (1.6). That restricts the eligible setups to intrinsically stationary (i.e., whose time-dependence, if it exists, can be gauged away by a change of frame) fields, and to a special class of observers therein; for electromagnetic fields in flat spacetime, those observers must be *static* in the inertial frame where the fields are explicitly time-independent. This is an important point that is worth discussing with some detail. Consider

the two basic analogous fields, Coulomb field of a point charge, and Schwarzschild gravitational field. Consider also in the latter observers \mathcal{O} in circular motion: 4-velocity $U^\alpha = (U^0, 0, 0, U^\phi)$, angular velocity $\Omega = U^\phi/U^0$; the worldlines of these observers are tangent to killing vectors: $U^\alpha \parallel \xi^\alpha$; $\mathcal{L}_\xi g_{\alpha\beta} = 0$; hence one uses to say (e.g. [28, 67]) that they see a constant spacetime geometry; for this reason they are called “stationary observers”. Now consider observers in circular motion around a Coulomb charge. Despite moving along worldlines tangent to vector which is a symmetry of the electromagnetic field: $U^\alpha \parallel \xi^\alpha$; $\mathcal{L}_\xi F_{\alpha\beta} = 0$, the observers U^α do *not* see a covariantly constant field: $F_{\alpha\beta;\gamma}U^\gamma \neq 0$ (they see a constant magnetic field \vec{B} , and an electric field \vec{E} constant in magnitude but varying in direction), which by virtue of Eqs. (1.6a), (1.4a), implies that the electromagnetic tidal tensors have an antisymmetric part (in particular the spatial part $B_{[ij]} \neq 0$), and thus means that they cannot be similar to their gravitational counterparts. This is a natural consequence of Maxwell’s equations and can be easily understood as follows. The magnetic tidal tensor measured by \mathcal{O} is a covariant derivative of the magnetic field as measured in the inertial frame momentarily comoving with \mathcal{O} : $B_{\alpha\beta} \equiv \star F_{\alpha\gamma;\beta}U^\gamma = B_{\alpha;\beta}|_{U=const.} = (B_{MCRF})_{\alpha;\beta}$. Now, $B_{[ij]} \neq 0$ means that \vec{B}_{MCRF} has a curl; which is natural since in the MCRF the electric field is time-dependent, which, by virtue of Maxwell equation $\nabla \times \vec{B} = \partial \vec{E}/\partial t = \gamma \vec{E} \times \vec{\Omega}$ (holding in the MCRF, and for which (1.6a) is a covariant form) induces a curl in \vec{B} .

But even if one considers static observers in stationary fields, so that the GR and EM tidal tensors have the same symmetries, still it does not mean a close similarity between the interactions. The electromagnetic tidal tensors are essentially derivatives of the electromagnetic fields, being themselves differential fields *linear* in the electromagnetic 4-potential $A^\alpha = (\phi, \vec{A})$, whereas the gravitational ones are *non-linear* in the GEM fields, as shown by Eqs. (87)-(88), being the gravitomagnetic field \vec{H} itself *non-linear* in the metric potentials Φ, \vec{A} . This means that one expects a similarity between tidal tensors in the two limiting cases — linearized theory, and the ultrastationary spacetimes considered in Sec. 100, where $\Phi = \vec{G} = 0$, and, therefore, cf. Eqs. (88) and (50), the exact gravitomagnetic tidal tensor is linear: $\mathbb{H}_{ij} = \tilde{D}_{\tilde{j}}(\tilde{\nabla} \times \vec{A})_{\tilde{i}}/2$. We have seen in Sec. 100 that there is indeed an exact mapping (via Klein-Gordon equation) between the dynamics in these spacetimes and an electromagnetic setup.

To what concerns the concrete effects, the precise conditions (namely regarding the time dependence of the fields) for occurrence of a gravito-electromagnetic similarity are specific to the type of effect (see also [25]). For the covariant effects (implying physical gravitational forces, i.e., tidal forces) such as the force on a spinning particle gyroscope or the worldline deviation of two neighboring particles, are the tidal tensors *as measured by the test particles* of 4-velocity U^α that determine the effects, cf. Eqs. (1.1)-(1.2); which means that *it is along the particle’s worldline that the time independence is required*. That basically implies that the similarity only occurs in the instant when the particles are at rest in stationary fields, so it does not hold in a truly dynamical situation. For the case of the correspondence between the Lorentz force Eq. (46), and the geodesic equation formulated as an inertial force, which is a *coordinate* effect, from Eq. (45) we see that the requirement is that the frame is rigid, i.e. $\sigma_{\alpha\beta} = \theta = 0$; as explained in Sec. 3.2, that amounts to say that the spatial part of the metric (in the coordinates associated to such frame) must be *time-independent*. This can also be stated in the following manner, generalizing to the exact case the conclusion obtained in [25] (in the context of the Post-Newtonian approximation): in the case of the GEM analogy for the geodesic equation, the *stationarity of the fields is required in the ob-*

server's frame (not in test particle's frame! The test particles can move along arbitrary worldlines). As for the gyroscope “precession” (52) and the correspondence with the precession of a magnetic dipole (53), there is no restriction on the time dependence of the fields.

8 Conclusion

In this work we collected and further developed different gravito-electromagnetic analogies existing in the literature, and clarified the connection between them. A detailed summary of the material in this paper is given in the introduction; we conclude by briefly summarizing the main outcome of each approach, and their applicability. The analogies split in two classes: physical and purely formal. In the second category falls the analogy between the electric and magnetic parts of the Weyl and Maxwell tensors, discussed in Sec 6. The physical analogies divide in two classes: exact analogies, and the best known Post-Newtonian and linearized theory approaches. Exact physical analogies are the analogy between the electromagnetic fields and the inertial fields of Sec. 3, and the tidal tensor analogy of Sec. 2.

These analogies are useful from a practical point of view, as they provide a familiar formalism, and insight from electromagnetic phenomena to describe otherwise more complicated gravitational problems. Indeed, there is a number of fundamental equations, summarized in Table 5, which can be obtained from the electromagnetic counterparts by simple application of the analogy. But the existence of these analogies, especially the exact, physical ones, is also interesting from the theoretical point of view, with intriguing similarities — both in the tidal tensor, and in the inertial field formalism, manifest in Tables 1 and 3, respectively — and enlightening differences unveiled (namely the ones manifest in the symmetries of the tidal tensors). The deep connection between gravitation and electrodynamics is still very much an open question; the similarities and fundamental differences unveiled in the analogies and formalism herein may be a small step in the direction of that understanding.

The tidal tensor formalism is primarily suited for a transparent comparison between the two interactions, since it is based on mathematical objects describing covariant *physical* forces common to both theories. The most natural application of the analogy is the dynamics of spinning multipole test particles, which is studied in the companion paper [4] to quadrupole order. As a formalism, consisting of mathematical objects with direct physical interpretation, encoding the gravitational *physical*, covariant forces, it can be useful in many applications, namely gravitational radiation (as discussed in Sec. 6.1), and whenever one wishes to study the physical aspects of spacetime curvature (as a further example, we mention the physical significance of the curvature invariants, which have attracted attention in the context of experimental astrophysics, and are studied in this framework in [63]).

The analogy based on inertial GEM fields from the 1+3 formalism, Sec. 3, is a very powerful formalism, with vast applications; especially in the case of stationary spacetimes, where for arbitrarily strong fields the equation for geodesics is cast in a form similar to Lorentz force; many other effects related to frame dragging can be treated exactly with the GEM fields: gyroscope “precession” (relative to the “distant stars”) [4, 19, 20, 40, 14], the gravitomagnetic clock effect [PUT REF], the Sagnac effect [112], the Faraday rotation [94], the force on a gyroscope (Sec. 3.6 and [14]; note however that it is not as general as the the tidal tensor formulation of the same force); and other

applications, such as the matching of stationary solutions [16], or describing the hidden momentum of spinning particles [4]. The general formulation of GEM fields in Sec. 3, applying to arbitrary fields and frames, extends the realm of applicability of this formalism.

Table 5: What can be computed by direct application of the GEM analogies

Result	Approach
<ul style="list-style-type: none"> • Geodesic deviation equation (1.1b): -Replacing $\{q, E_{\alpha\beta}\} \rightarrow \{m, -\mathbb{E}_{\alpha\beta}\}$ in (1.1a). • Force on a gyroscope (1.1b): -Replacing $\{\mu^\alpha, B_{\alpha\beta}\} \rightarrow \{S^\alpha, -\mathbb{H}_{\alpha\beta}\}$ in (1.1a). • Differential Precession (1.8b): -Replacing $\{\sigma, B_{\alpha\beta}\} \rightarrow \{1, -\mathbb{H}_{\alpha\beta}\}$ in (1.1a). • Temporal part of Einstein's equations (1.3b) -(1.6b): -Replacing $\{E_{\alpha\beta}, B_{\alpha\beta}\} \rightarrow \{\mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta}\}$ in Eqs. (9)-(12), and $\rho_c \rightarrow 2\rho + T^\alpha_\alpha$ in (9), $j^\alpha \rightarrow 2J^\alpha$ in (12). 	<p>Tidal tensor analogy</p> <p>(Exact, general results)</p>
<ul style="list-style-type: none"> • Geodesic Equation (48) (<i>stationary</i> fields) -Replacing $\{\vec{E}, \vec{B}\} \rightarrow \{\vec{G}, \vec{H}\}$ in (46), multiplying by γ. • Gyroscope “precession” Eq. (52) (<i>arbitrary</i> fields): -Replacing $\{\vec{\mu}, \vec{B}\} \rightarrow \{\vec{S}, \vec{H}/2\}$ in (53). • Force on gyroscope Eq. (98) (<i>stationary</i> fields, particle's worldline tangent to time-like Killing vector): -Replacing $\{\vec{\mu}, \vec{E}, \vec{B}\} \rightarrow \{\vec{S}, \vec{G}, \vec{H}/2\}$ in (96), factor of 2 in the last term. 	<p>Inertial “GEM fields” analogy</p> <p>(Exact results, require special frames)</p>
<ul style="list-style-type: none"> • Higher order field equations (4.1b)-(4.4b): -Replacing $\{\vec{E}, \vec{B}\} \rightarrow \{\mathbb{E}_{ij}, \mathbb{H}_{ij}\}$ in Eqs. (4.1a)-(4.4a). • Equations of gravitational waves (4.5b)-(4.6b): -Replacing $\{\vec{E}, \vec{B}\} \rightarrow \{\mathbb{E}_{ij}, \mathbb{H}_{ij}\}$ in Eqs. (4.5a)-(4.6a). 	<p>Weyl-Maxwell tensors analogy</p> <p>(Results for linearized theory)</p>

The well known analogies between electromagnetism and Post-Newtonian, and linearized gravity, follow as a limiting case of the exact approach in Sec. 3. In the case of the tidal effects, they can be seen also as limiting case of the tidal tensor analogy of Sec. 2 (in the sense that for weak, *time-independent fields*, the tidal tensors are derivatives of the GEM fields). Acknowledging this fact, and understanding the conditions under which linear GEM is obtained from the rigorous, exact approaches, is important for a correct interpretation of the physical meaning of the quantities involved, which is not clear in the usual derivations in the literature (this is especially the case for many works on linear GEM), and thus prone to misconceptions [2, 1]. On the other hand, linear

GEM is the most important in the context of experimental physics, as it pertains all gravitomagnetic effects detected to date [41, 106, 42, 116, 115, 114], and the ones we hope to detect in the near future [43].

As for the analogy between the electric and magnetic parts of the Weyl and Maxwell tensors, its most important application is gravitational radiation, where it provides equations for the propagation of *tensors of physical forces* (not components of the metric tensor, as in the usual approaches, which are pure gauge fields), with direct translation in physical effects via the tidal tensor formalism of Sec. 2. This analogy has also been used to address the fundamental questions of the content of gravitational waves, and the energy of the gravitational field. Namely, to define covariant, local quantities alternative to the gravitational energy and momentum given by the Landau-Lifshitz pseudo-tensor (which can only have a meaning in a global sense, and in asymptotically flat space-times): the super-energy and super-momentum encoded in the Bel tensor. The motivation for the definition of this tensor is the analogy with electromagnetism; and the existing criteria for radiative states [110], states of intrinsic radiation [31, 109] or pure radiation ([113], see also [60] p. 53), are also solely driven by it. The analogy is also useful for the understanding of the quadratic invariants of the curvature tensor; indeed, it will be shown elsewhere [63] that using the two approaches together — the formal analogy between the Weyl and Maxwell tensors to gain insight into the invariant structure, and the tidal tensor analogy as a physical guiding principle — one can explain, in the astrophysical applications of present experimental interest (as mentioned above), the significance of the curvature invariants and the implications on the motion of test particles.

Acknowledgments

We thank J. Penedones for useful comments and remarks; we also thank A. García-Parrado for correspondence and useful discussions.

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